OXFORD IB COURSE PREPARATION

MATHEMATICS

FOR IB DIPLOMA COURSE PREPARATION

Jim Fensom

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Worked solutions can be found at **www.oxfordsecondary.com/9781382004923**

Introduction

The Diploma Programme (DP) is a two-year pre-university course for students in the 16–19 age group. In addition to offering a broad-based education and in-depth understanding of selected subjects, the course has a strong emphasis on developing intercultural competence, open-mindedness, communication skills and the ability to respect diverse points of view.

You may be reading this book during the first few months of the Diploma Programme or working through the book as a preparation for the course. You could be reading it to help you decide whether the Maths course is for you. Whatever your reasons, the book acts as a bridge from your earlier studies to DP Maths, to support your learning as you take on the challenge of the last stage of your school education.

DP course structure

The DP covers six academic areas, including languages and literature, humanities and social sciences, mathematics, natural sciences and creative arts. Within each area, you can choose one or two disciplines that are of particular interest to you and that you intend to study further at the university level. Typically, three subjects are studied at higher level (HL, 240 teaching hours per subject) and the other three at standard level (SL, 150 hours).

In addition to the selected subjects, all DP students must complete three core elements of the course: theory of knowledge, extended essay, and creativity, action, service.

Theory of knowledge (approximately 100 teaching hours) is focused on critical thinking and introduces you to the nature, structure and limitations of knowledge. An important goal of theory of knowledge is to establish links between different areas of shared and personal knowledge and make you more aware of how your own perspective might differ from those of others.

The **extended essay** is a structured and formally presented piece of writing of you to 4,000 words based on independent research in one of the approved DP disciplines. It is also possible to write an interdisciplinary extended essay that covers two DP subjects. One purpose of the extended essay activity is to develop the high-level research and writing skills expected at university.

Creativity, action, service involves a broad range of activities (typically 3–4 hours per week) that help you discover your own identity, adopt the ethical principles of the IB and become a responsible member of your community. These goals are achieved through participation in arts and creative thinking (creativity), physical exercises (activity) and voluntary work (service).

DP Mathematics syllabus

Basics

Two Mathematics subjects are offered in Group 5 of the IB Diploma: Mathematics: analysis and approaches (MAA) and Mathematics: applications and interpretation (MAI). For your IB Diploma course you will need to select one Mathematics course from:

- Mathematics: Analysis and Approaches (MAA) Standard Level (SL)
- Mathematics: Analysis and Approaches (MAA) Higher Level (HL)
- Mathematics: Applications and Interpretation (MAI) Standard Level (SL)
- Mathematics: Applications and Interpretation (MAI) Higher Level (HL)

The whole content of each SL course forms part of the corresponding HL course. There is also considerable overlap in the content of MAA and MAI and, indeed, 60 teaching hours is common to all four syllabuses.

One of the purposes of this book is to help guide you to the most appropriate course for you.

The courses

The MAA course follows a traditional approach to High School mathematics. The course guide describes a typical MAA student as one who "should be comfortable in the manipulation of algebraic expressions and enjoys the recognition of patterns and understands the mathematical generalization of these patterns … and (will have) the ability to understand simple proof." (IB Mathematics course guide 2019)

The MAI course puts more emphasis on the mathematics used in the workplace or in those other academic disciplines which increasingly rely on mathematics to underpin the work they do – for example, Biology, Environmental science, Economics, Medicine, Sociology. The course guide describes a typical MAI student as one who "should enjoy seeing mathematics used in real-world contexts and being used to solve real-world problems." (IB Mathematics course guide 2019)

How the book will help you to choose a course

This book covers the prior learning required for all the courses and the questions will help you develop the necessary skills and techniques for whichever course you do. There are a few sections in the book that are based on prior learning required for the HL courses. These are clearly labeled as "Higher level". You do not need to cover these sections if you are preparing for an SL course.

To help you decide which of the two subjects – MAA or MAI – might suit you best, some questions are labeled as **DP Style: Analysis and approaches** or **DP Style: Applications and interpretation**. There is a lot of overlap in the style as well as the content of the two courses, but the questions here focus on the differences.

The DP style: MAA questions will often allow you to work at a slightly more abstract level, to use algebra to generalize results, to make conjectures and devise simple proofs.

The DP style: MAI questions are mainly set in a real-world context. You will need to decide how to use the mathematics you have learned to solve the real-world problem. Then, when you have worked out the mathematics, you will need interpret your answer within the context given in the question.

Both of the higher level courses, and both of the standard level courses, are seen by the IB as equally challenging. There is, however, a considerable increase in difficulty between each standard level course and the corresponding higher level course. This is an increase in the difficulty of the concepts covered, as well as in the complexity of the mathematics and the depth of understanding required to succeed on the course.

In order to help you to decide whether or not a higher level course is for you, the DP-style questions are labeled as either SL or HL. These are questions based on the work within the prior learning section of the syllabus, but they are also designed to help you to develop these ideas and allow you to explore some of the consequences of the mathematics.

A potential higher level candidate is not expected to complete all of these DP style HL questions easily, but they should enjoy trying them. The IB guide describes a potential higher level student as someone who "enjoys spending time with problems and who gets pleasure and satisfaction from (their solution)".

Each chapter ends with a test. The test includes a DP-style question for both of the courses and for each of the levels.

Your experience of tackling the different types of question will indicate which course would be the most rewarding for you. This though is only one of several factors in deciding. You also need to consider which courses are offered by your school, which other diploma courses you are taking and any entry requirements for particular universities.

Internal Assessment

All DP subjects have an Internal Assessment (IA) component. This assessment allows students to demonstrate their knowledge and to apply skills in a topic of choice often linked to personal interest. The exploration is a piece of work that involves in-depth investigation in a chosen area of mathematics and /or its application to a real-world problem.

Although the IA has no time limitation like externally-assessed components, schools will have a timeline and internal deadlines for submission of a draft and the final submission after written feedback has been given on the draft. It is internally assessed by the teacher against five criteria, but like all DP internal assessment components it is subject to external moderation.

Using this book effectively

Throughout this book you will encounter separate text boxes to alert you to ideas and concepts. Here is an overview of these features and their icons:

Icon	Feature	Description of feature
	Key terms	Mathematical terms that you need to learn to prepare you for the DP Mathematics course.
	Key point	Definition or rule.
	Investigation	A mathematical investigation, with step-by-step instructions. This will help develop your mathematical understanding of a topic and your inquiry skills. It also prepares you for your DP Mathematics course, which is taught through investigations.
	Worked example	A question with a full worked solution. The working and answer are in the left-hand column. Notes in the right-hand column explain the steps in the working.
	Exercise	Questions for you to answer, to practise what you have learned.
	Note	Extra information to help you understand a worked example or an explanation.
Hint	Hint	A hint to help you answer a mathematics question.
	Command term	Explanation of a command term – a word that tells you what you need to do in a question, for example **identify**, or **describe**.
DP style	DP style question	A question based on the mathematics in this book, written in the style of DP questions. Labelled MAI or MAA, SL or HL, to help you get an idea of the type and difficulty of questions you will be working on in each course.
Higher Level	Higher level content	Mathematics that is only required for the Higher level DP Mathematics courses.
	Internal link	Reference to another section in this book, where there is more information on a topic.
	DP link	Explanation of how this topic will be used or developed further in the DP courses.
	DP ready: International-mindedness	Description of the use of mathematics around the world.
	DP ready: Theory of knowledge	Ideas or concepts in mathematics that prompt wider discussions about the different ways of knowing.
	DP ready: Approaches to learning	Lists the skills you need to be an effective DP Mathematics learner, and that you will develop as you work through the activity.

You may have covered some of the mathematics in this book before, so you may find you do not need to spend equal amounts of time on each of the chapters.

A good place to start is the first page of each chapter where the learning outcomes and key terms to be covered are listed. You can also check the chapter summary, which comes immediately before the end-of-chapter test.

NUMBER AND ALGEBRA

1 Number

Learning outcomes

In this chapter you will learn about:

- Number systems
- Sets
- Approximation and rounding
- Absolute value
- Operations with numbers
- Prime numbers, factors and multiples
- Greatest common factor and least common multiple (HL only)
- Exponential expressions
- Exponential expressions with rational indices (HL only for MAI, SL for MAA)
- Surds (radicals)
- Rationalising the denominator (HL only)
- Standard form

Key terms

- Set
- Approximate
- Modulus
- Prime number
- Multiple
- Factor (divisor)
- Fraction
- Exponent (also called power or index)
- Surd (radical)
- Standard form

1.1 Number systems

Key point

The natural numbers, $\mathbb{N}$, are the counting numbers (including the number 0): 0, 1, 2, 3, 4, 5, …

The integers, $\mathbb{Z}$, include all the natural numbers and their negative values too: … −2, −1, 0, 1, 2, …

The rational numbers, $\mathbb{Q}$, include all the integers and numbers in the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers. Examples of rational numbers are $\frac{5}{6}, -\frac{1}{2}, \frac{20}{7}, 3\frac{1}{2}, \ldots$

Terminating decimals like 0.5, 175, and −7.396 and recurring decimals like 0.333… and 1.7272… are also examples of rational numbers, since they can be written in the form $\frac{a}{b}$, $b \neq 0$.

The real numbers, $\mathbb{R}$, contain both the rational and irrational numbers. Irrational numbers include π, $\sqrt{2}$, $\sqrt[3]{10}$ and any non-terminating, non-recurring decimals.

DP ready — International-mindedness

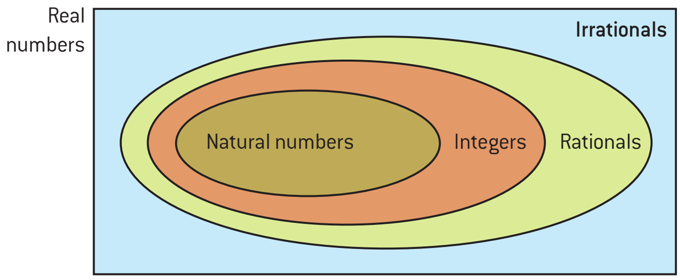
Most early number systems, such as the Babylonian, Egyptian and Roman systems, had no symbol for zero. There is evidence that the Mayan civilisation in Central America had a zero symbol in the first century BCE, the Khmer civilisation in Cambodia used a zero symbol in the 6th century CE and the Hindu number system in India used a symbol for zero in the 9th century CE.

The Western number system, based on the Hindu-Arabic system, was only introduced in the 13th century CE.

DP link

In Mathematics DP, HL students also learn about a set of numbers called the complex numbers, $\mathbb{C}$.

These groups of numbers can be organised in sequence, where each group contains all of the numbers from the previous group(s): natural numbers ($\mathbb{N}$); integers ($\mathbb{Z}$); rationals ($\mathbb{Q}$) and irrationals; real numbers ($\mathbb{R}$). For example, the rational numbers contain all of the integers and all of the natural numbers.



Command term

Identify means you should choose an answer from a number of possibilities.

Describe means you should give a detailed account.

Exercise 1.1

1 **Identify** which is the *smallest* set ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$) each number belongs to.

1 12 **2** -4 **3** 0 **4** $\sqrt{5}$

5 $\sqrt{16}$ **6** $\frac{5}{7}$ **7** $-\frac{12}{4}$

1.2 Sets

Key point

A **set** is a collection of objects: for example, a collection of numbers, letters, geometrical objects or anything else.

As you saw in 1.1, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are sets of numbers.

You **describe** a set by writing a list of everything in it, or by writing a description of what is contained in the set, written inside curly brackets (or braces) { }.

For example:

$A = \{1, 2, 3, 4\}$

$B = \{a, e, i, o, u\}$

$C = \{red, green, blue\}$

$D = \{English, Chinese, History, Physics, Mathematics, Art\}$

$E = \{$Integers between 1 and 10$\}$

$F = \{$Countries in the EU$\}$

$G = \{$Planets in the Solar System$\}$

$H = \{$Irrational numbers$\}$

Key point

The number of elements in a set A is its **cardinality**, $n(A)$.

The set that contains no elements at all is the **empty set**, $\varnothing$. $n(\varnothing) = 0$.

The set of all the elements you are considering is the **universal set**, U.

You call individual items in a set **elements**.

Key point

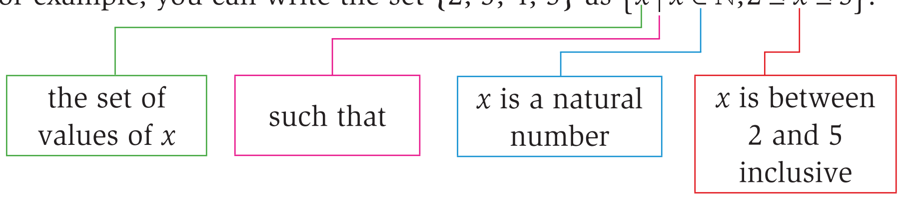
The symbol $\in$ means 'is an element of' and $\notin$ means 'is not an element of'.

For example:

$2 \in \{1, 2, 3, 4\}$ Jupiter $\in$ {Planets in the Solar System}

You can also describe sets using **set-builder notation**.

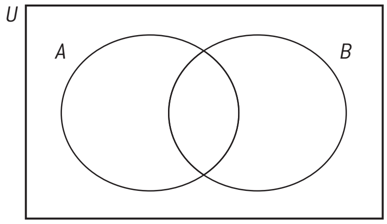
For example, you can write the set {2, 3, 4, 5} as $\{x \mid x \in \mathbb{N}, 2 \le x \le 5\}$.



Internal link

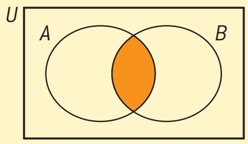
$2 \le x \le 5$ is an example of an inequality. You will study these in chapter 2.

A **Venn diagram** is a way of representing sets:

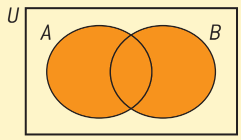


Key point

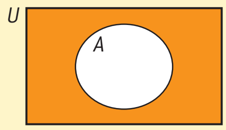
The set of elements that are in both A and B is the **intersection** of A and B, $A \cap B$.



The set of elements that are either in A or B or both is the **union** of A and B, $A \cup B$.

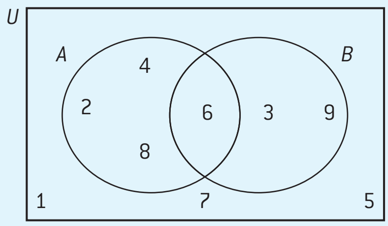


The set of elements of U that are not in A is the **complement** of A, that is, A'. $A' = \{x \mid x \in U, x \notin A\}$.



Example 1

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{3, 6, 9\}$.
Draw a Venn diagram to represent these sets.



6 is in both A and B so write it in the intersection.

The other elements of A are 2, 4 and 8. Write these in the outer part of A.

The other elements of B are 3 and 9. Write these in the outer part of B.

The other elements of U are 1, 5 and 7. Write these outside of A and B.

DP link

DP students use Venn diagrams to tackle problems in probability in MAI and MAA at HL and SL.

Exercise 1.2

1 **Write down** the set of natural numbers that are less than 6:

a as a list inside braces b using set-builder notation.

2 Consider the sets $P = \{2, 4, 6, 8\}$, $Q = \{1, 3, 5, 7\}$, $S = \{x \mid x \in \mathbb{Z}, 2 < x \le 6\}$.

Determine which of the following statements are true and which are false. Where a statement is false, re-write and correct it so that it is true.

a $4 \in P$ **b** $5 \notin Q$ **c** $P \cap Q = \varnothing$ **d** $n(P) = 4$ **e** $n(P) = n(S)$

DP style Analysis and Approaches HL

3 $U = \{n \mid n \in \mathbb{N}, n \le 10\}$, $M = \{1,2,3,5,8\}$ and $N = \{3,6,8\}$

a Draw a Venn diagram to show sets U, M and N. Write the numbers 0 to 10 in the appropriate section of the diagram.

Write down each set as a list:

b M' **c** N' **d** $M \cap N$ **e** $M \cup N$

Command term

Write down means you should obtain the answer without any calculations. You do not need to show any working.

Determine means you should obtain the only possible answer.

Note

To help you decide which of the two routes – Analysis and approaches or Applications and interpretation – might suit you best, questions have been labelled throughout. You are not expected to complete all of these DP-style questions easily, but working through them should help you to decide where your interests lie!

4 Write down the definition of rational numbers in 1.1 using set-builder notation.

DP style **Analysis and Approaches HL**

5 Given that $n(A) = 6$, $n(B) = 12$, $n(A' \cap B) = 9$ and the universal set has 24 elements, write down:

a $n(A \cap B)$ **b** $n(A' \cap B')$ **c** $n(A' \cup B')$.

DP style **Applications and Interpretation HL**

6 A student conducts a survey of cars that pass the school. She notes the colour and the make of the cars. When looking at the data she notices the most common colour is silver and the most common make is Peugeot.

Of the 100 cars she surveyed, 38 were silver and 22 were made by Peugeot.

Given 48 were either made by Peugeot or were silver, use a Venn diagram to find:

a the number of cars that were neither made by Peugeot, nor were silver

b the number of silver Peugeots.

1.3 Approximations and rounding

When you use a calculator, the result may be more accurate than you need.

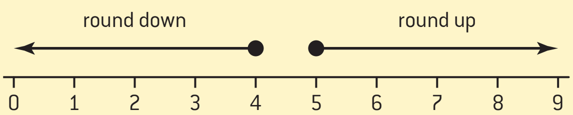
For example, Adam earns € 35 023 per year. He calculates his monthly salary:

$$35\,023 \div 12 = 2918.58333\ldots$$

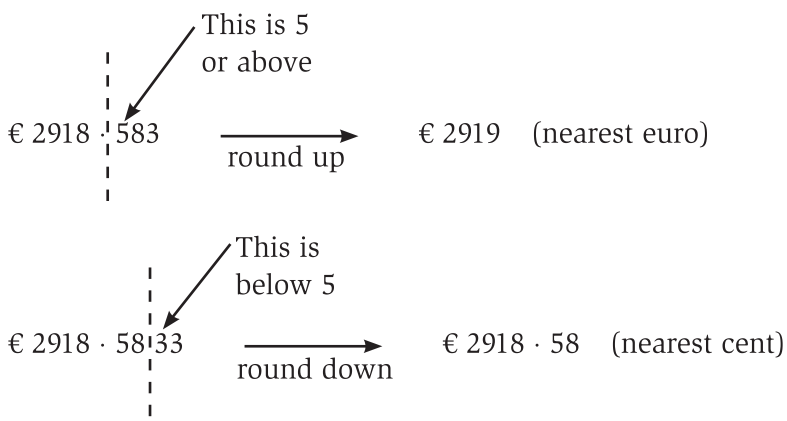
He can round this to the nearest whole number of euros or he can round to 2 decimal places (d.p.) so it is in euros and cents.

Key point

When rounding, consider the figure immediately to the right of the last digit you are rounding to. If the next figure is 0, 1, 2, 3, or 4, round down. If the next figure is 5, 6, 7, 8, or 9, round up.



For Adam,



Example 2

1 Rahul earns € 35 060 per year.

a **Calculate** his monthly salary.

b Calculate his weekly pay.

2 Write down the value of 456.8 to the nearest whole number.

1 a $35\,060 \div 12 = 2921.666\ldots$ Rounding to 2 d.p. Rahul's monthly salary is € 2921.67	Round the amount in euros to the nearest cent. Look at the 3rd decimal place. Since it is 6, you round up the 2nd decimal place.
b $35\,060 \div 52 = 674.2307\ldots$ Rounding to 2 d.p. Rahul's weekly salary is € 674.23	There are 52 weeks in a year, so divide by 52. Round the amount in euros to the nearest cent. Look at the 3rd decimal place. Since it is 0, you round down the 2nd decimal place.
2 456.8 is 457 to the nearest whole number.	Look at the 1st decimal place. Since it is 8, you round the number in the units column from 6 up to 7.

> **Command term**
>
> **Calculate** means you should obtain a numerical answer showing the relevant stages in your working.

In IB examinations you should give numerical answers exactly or to three significant figures.

The first significant figure is the first non-zero digit from the left.

For example, 2.1538461538 = 2.15 to 3 s.f., 0.0215386 = 0.0215 to 3 s.f. and 40.52 = 40.5 to 3 s.f.

Example 3

Round these numbers to 3 s.f.

a 12.72 **b** 10 730 **c** 0.02646 **d** 34.65 **e** 7895

a 12.72 = 12.7 to 3 s.f.	Look at the 4th figure. Since it is 2, you round the 3rd figure down.
b 10 730 = 10 700 to 3 s.f.	The 4th figure is 3, so you round the 3rd figure down. Insert a zero to keep the place value.
c 0.02646 = 0.0265 to 3 s.f.	Begin counting from the first non-zero figure. The 4th figure is 6, so you round the 3rd figure up.
d 34.65 = 34.7 to 3 s.f.	The 4th figure is 5, so you round the 3rd figure up.
e 7895 = 7900 to 3 s.f.	The 4th figure is 5 so you would round the 3rd figure up, but since it is 9, you round up by 'adding 1' to the 3rd digit, which makes it ten.

> **Hint**
>
> If you are continuing a calculation and using an answer further, do not round your first answer. Only when you reach the final answer should you round to an appropriate degree of accuracy.

DP link

Consideration of errors and the effect they have on real-life situations is one of the themes of the MAI course.

Command term

Comment means you should make an observation based on the result of your calculation.

Calculator hint

Your calculator has a key marked Ans. You can press this to use the most recently calculated value to the full accuracy of your calculator. As a shortcut, for example, instead of pressing the Ans key and typing × 5, you can type × 5 and the calculator will insert Ans for you. Some calculators also let you copy and paste answers. All calculators let you store values to use later in a calculation.

Using the value to the full accuracy of your calculator makes sure that your final answer is as accurate as possible.

Exercise 1.3

1 Round each of the following numbers given in parts **a** – **i** to

- **i** 2 d.p.
- **ii** the nearest whole number
- **iii** 3 s.f.

a	764.382	**b**	234.368	**c**	0.02379
d	0.005456	**e**	15.098	**f**	86.798
g	178867.352	**h**	0.5798	**i**	29.891

DP style Applications and Interpretation HL

2 A box contains 13 nails and costs \$2.90.

a The cost of the box goes up by \$0.50. Calculate the new price of one nail. Write down the answer given by your calculator and then round it to 3 s.f.

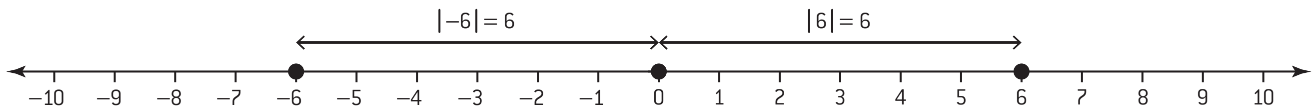
b The shopkeeper decides to put 15 nails in each box instead of 13. Use the new price of one nail from part **a** which is

- **i** given unrounded by your calculator
- **ii** rounded to 3 s.f.

to calculate the price of a box of 15 nails, correct to 2 d.p. **Comment** on what your answers to part **b** tell you about rounding before the final answer.

1.4 Modulus (or absolute value)

The modulus tells you how far away a number is from zero. It does not matter whether the number is positive or negative, so (for example) $|-6| = 6$ and $|6| = 6$.



Key point

The **modulus** or absolute value $|x|$ of x is $\begin{cases} x \text{ for } x \geq 0 \\ -x \text{ for } x < 0 \end{cases}$.

Investigation 1.1

Copy and complete the table.

a	b	$\lvert a+b\rvert$	$\lvert a\rvert+\lvert b\rvert$	$\lvert a-b\rvert$	$\lvert b-a\rvert$	$\lvert a\rvert-\lvert b\rvert$	$\lvert\lvert a\rvert-\lvert b\rvert\rvert$	$\lvert a\times b\rvert$	$\lvert a\rvert\times\lvert b\rvert$
2	5								
−3	4								
1	−2								
−5	−4								

1 Look at the results for $|a+b|$ and $|a|+|b|$. What do you notice?

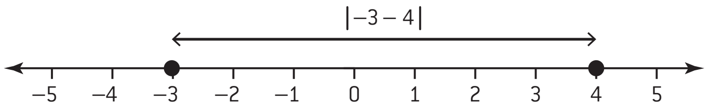
Use one of the symbols =, ≥ or ≤ (where ≤ means *less than or equal to*, and ≥ means *greater than or equal to*) to replace □ and complete the following conjecture.

$|a+b| \,\square\, |a|+|b|$

2 Look at the results for $|a-b|$ and $|b-a|$. What do you notice?

Use one of the symbols =, ≤ or ≥ to replace □ and complete the following conjecture.

$|a-b| \,\square\, |b-a|$



This distance is called the **absolute difference** of the two numbers.

3 Look at the results for $|a-b|$ and $||a|-|b||$. What do you notice?

Use one of the symbols =, ≤ or ≥ to replace □ and complete the following conjecture.

$|a-b| \,\square\, ||a|-|b||$

4 Look at the results for $|a\times b|$ and $|a|\times|b|$. What do you notice?

Use one of the symbols =, ≤ or ≥ to replace □ and complete the following conjecture.

$|a\times b| \,\square\, |a|\times|b|$

Exercise 1.4

1 If $a=-5$, $b=3$ and $c=-2$, **find**:

a $|ab|$ **b** $|bc|$ **c** $|abc|$ **d** $|a|\times|bc|$ **e** $|a|+|b|-|c|$ **f** $|ab|-|bc|$

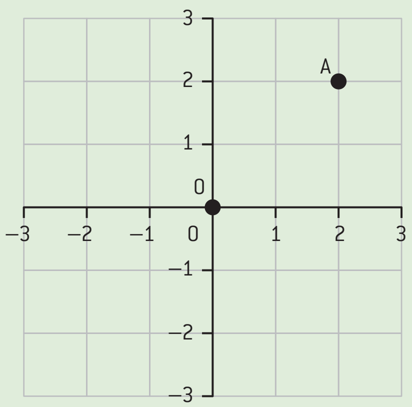
2 If $p=4$ and $q=-10$, find:

a $\left|\frac{p}{q}\right|$ **b** $\left|\frac{q}{p}\right|$ **c** $\frac{|p|}{|q|}$ **d** $\frac{|q|}{|p|}$ **e** $|pq^2|$ **f** $\frac{p}{|q^2|}$

DP style MAA and MAI HL

3 The symbol $|a|$ is a measure of the shortest distance from a to zero.

Imagine a city built on a grid, with roads running north–south and east–west along every unit square. To get from one place to another, you can only travel along the roads; you cannot travel through the middle of a square. The point O is at the centre of the grid.



Command term

Find means you should obtain an answer, showing relevant stages in the working.

DP link

The MAA course will look at a range of proofs in mathematics. A result that has been proved is stronger than a conjecture as you are showing that the result is true in all cases, rather than just the specific examples from your investigation. Those at HL are more rigorous than at SL. Conjecture and proof are part of MAA HL paper 3 in particular.

Command term

Verify means

DP link

This question follows the HL paper 3 idea (for both the MAA and MAI courses) of taking an idea and extending it with further questions.

If A is a point on the grid, let |OA| be the shortest distance of travelling from O to A.

a If A is at (2,2), find |OA|.

b Conjecture a formula for |OA| if A is at the point (a,b).

c **Verify** your formula is true for the point A (−2,1).

In this system we define a **circle** with radius of 4 and centre (0,0) as the set of points which are a distance of 4 units from (O,O). Using this definition, a circle of radius 4 on the grid is the set of all points, A, for which $|OA| = 4$.

d Draw the circle of radius 4 on graph paper. Note: this will be a series of points rather than a continuous curve.

The diameter of a circle is defined as the longest distance between any two points on the circle.

e Verify that the diameter (d) of the circle is 2 times the radius.

The circumference (C) of the circle is the distance around all the points in the circle.

f Given that $C = pd$, find the value of p and verify this value is the same when considering a circle of a different radius.

DP ready **Theory of knowledge**

In investigations you look at a range of specific examples and then, from these examples, you try to deduce a general result. This result is called a **conjecture**.

How do you know that a conjecture is always true?

Mathematicians try to **prove** a conjecture. In the 17th century, Pierre de Fermat made a conjecture that $x^n + y^n = z^n$ has no solutions where x, y and z are all integers for values of $n \geq 3$. For three centuries, mathematicians tried to prove or disprove this. Finally, in 1993, Andrew Wiles presented a proof of the theorem at a conference in Cambridge.

1.5 Operations with numbers

How can you find the value of $2 + 3 \times 4$?

Either: step (1): $2 + 3 = 5$ or step(1): $3 \times 4 = 12$

step (2): $5 \times 4 = 20$ step(2): $2 + 12 = 14$

Is the answer 20 or 14? It depends which order you carry out the operations.

The correct answer is 14. You get this when you carry out operations in the following way:

What happens when you add three numbers together? For example, $2 + 3 + 4$.

If you add 2 and 3 first, you get $(2 + 3) + 4 = 5 + 4 = 9$, but if you add 3 and 4 first then you get $2 + (3 + 4) = 2 + 7 = 9$, which is the same answer. So $2 + 3 + 4$ is not ambiguous. You do not need any parentheses to make it clear.

What happens when you multiply three numbers together? For example, $7 \times 2 \times 3$.

If you multiply 7 and 2 first, you get $(7 \times 2) \times 3 = 14 \times 3 = 42$, but if you multiply 2 and 3 first then you get $7 \times (2 \times 3) = 7 \times 6 = 42$, which is the same answer. So $7 \times 2 \times 3$ is not ambiguous. You do not need any parentheses to make it clear.

Multiplication and addition are examples of operations that are **associative**. The order you perform repeated operations that are associative makes no difference to the answer.

Does the same thing happen with subtraction? For example, is $(10 - 3) - 2$ the same as $10 - (3 - 2)$? Here $(10 - 3) - 2 = 7 - 2 = 5$ and $10 - (3 - 2) = 10 - 1 = 9$, so the answers are not the same.

Key point

You can summarize the rules for the order of operations as:

1. **B**rackets (or parentheses),
2. **O**rders (indices or exponents),
3. **M**ultiplication/**D**ivision,
4. **A**ddition/**S**ubtraction,

You can remember this using the mnemonic **BOMDAS**

Subtraction is not associative, so $10 - 3 - 2$ *is* ambiguous. Here you should work from left to right, but to make the calculation clear you should also include parentheses.

Division is not associative either. $(12 \div 6) \div 2 = 2 \div 2 = 1$ and $12 \div (6 \div 2) = 12 \div 3 = 4$. These answers are not equal, so you should use parentheses to make repeated division clear.

Example 4

Calculate **a** $3 + 6 + 12$ **b** $2 + 4 - 3$ **c** $8 + 2 \times 3$ **d** $13 - (5 - 2)$ **e** $5 \times 2 \times 7$ **f** $18 \div (2 \times 3)$

a $3 + 6 + 12 = 9 + 12$ $= 21$	Since + is associative you can also calculate $3 + 6 + 12 = 3 + 18 = 21$
b $2 + 4 - 3 = 6 - 3$ $= 3$	The correct order is from left to right.
c $8 + 2 \times 3 = 8 + 6$ $= 14$	You should multiply before you add.
d $13 - (5 - 2) = 13 - 3$ $= 10$	Calculate the subtraction inside the parentheses first.
e $5 \times 2 \times 7 = 10 \times 7$ $= 70$	Since × is associative the order of calculation makes no difference to the answer. It is slightly easier to multiply $5 \times 2 = 10$ first because it is easier to multiply by 10 than it would be to multiply by 14 if you had multiplied 2×7 first.
f $18 \div (2 \times 3) = 18 \div 6$ $= 3$	You can write a division using a fraction line. $18 \div (2 \times 3)$ is the same as $\frac{18}{2 \times 3}$. The fraction line takes the place of the parentheses. Take care with calculations like this.

Exercise 1.5

1 Calculate

a $3 + 4 - 2$ **b** $10 - 5 - 2$ **c** $30 \div 6 \times 2$ **d** $36 \div 4 \div 3$

2 Find

a $8 - 4 \div 2$ **b** $15 \times 2 + 4$ **c** $7 - 2 \times 3$ **d** $9 - 10 \div 5 + 2$

3 Calculate

a $6 \times (5 - 3)$ **b** $3(4 + 2) \div 2$ **c** $\frac{24}{3 \times 4}$ **d** $\frac{5 - 2 \times 3}{3 \times 2 - 7}$

When you use your calculator, entering calculations in the same way as you would write them on paper can help a lot. Your GDC will then use the correct order of operations.

Use the fraction template that looks like $\frac{\square}{\square}$ to enter a fraction line. Find out how to do this with your GDC.

Internal link

Return to **Exercise 1.5** and calculate each of the answers with a GDC. Check that you get the same results as when you calculated them by hand.

1.6 Prime numbers

When you learned multiplication tables, $1 \times 2 = 2, 2 \times 2 = 4, 3 \times 2 = 6, \ldots$ you were learning about multiples. The result of multiplying one positive integer by another is its **multiple**. So, the multiples of 2 are $2, 4, 6, \ldots$

The number 12 is a multiple of 1, 2, 3, 4, 6 and 12.

A **factor** (or **divisor**) is a positive integer that will divide exactly into another integer. So, 1, 2, 3, 4, 6 and 12 are all **factors** of 12.

Command term

List means you should give a sequence of brief answers with no explanation.

Key point

A **prime number** is a positive integer, greater than 1, that has exactly two factors. It is not a multiple of any other number apart from 1 and itself. Numbers that have more than two factors are composite. The number 1 is neither prime nor composite.

Example 5

a **List** the first 8 multiples of 3.

b Find all the factors of 48.

a The multiples of 3 are:

3, 6, 9, 12, 15, 18, 21, 24.

b $48 = 1 \times 48, 2 \times 24, 3 \times 16, 4 \times 12, 6 \times 8$

The factors of 48 are:

1, 2, 3, 4, 6, 8, 12, 16, 24, 48

Example 6

Which of these numbers is prime?

a 17 **b** 9 **c** 51 **d** –11 **e** 0

17 is prime	$17 = 17 \times 1$ are the only two factors
9 is not prime	$9 = 9 \times 1$ $9 = 3 \times 3$. It has more than two factors
51 is not prime	$51 = 51 \times 1$ $51 = 17 \times 3$
–11 is not prime	–11 is not positive
0 is not prime	0 is not a positive integer

Investigation 1.2

In this investigation, you will find all the prime numbers between 1 and 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1 Draw a circle around 2. Cross through all multiples of 2 (such as 4, 6, 8, 10,…).
2 3 is the next number that you have not marked. Draw a circle around 3 and cross through all of its multiples. You have already crossed some of these.
3 Continue through the table, circling the first unmarked number and crossing through all its multiples.
4 What is the largest circled number that has a multiple in the square?
5 Circle the remaining unmarked numbers that are greater than 1.
6 List all the circled numbers: 2, 3, … etc. These are the prime numbers.

The number 1 is still unmarked. It is neither a prime nor a composite number.

7 Find the values of $x^2 + x + 11$ when $x = 0, 1, 2, \ldots$ to complete this table

x	0	1	2	3	4	5	6	7	8	9	10
$x^2 + x + 11$	11	13									

8 The values when $x = 0$ and when $x = 1$ are both prime numbers. Which other values of x give prime values? Which is the smallest composite number?

9 Does the formula $x^2 + x + 11$ find all the prime numbers up to the first composite number?

Exercise 1.6

1 Are these numbers prime or composite?

a 113 **b** 251 **c** 119 **d** 173 **e** 169

DP style Analysis and Approaches

2 The operation $\cdot$ is defined for $a, b \in \mathbb{Z}$ by $a \cdot b = a + b - 2$

a By considering $a \cdot b \cdot c$ show that $\cdot$ is associative.

The identity element e is such that $a \cdot e = a$.

b Find the value of e.

c Find **i** $a \cdot a$ **ii** $a \cdot a \cdot a$ **iii** $a \cdot a \cdot a \cdot a$

d Suggest an expression for $\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a \cdot a \cdot a}_{n \text{ terms}}$

e Hence, find $a \cdot 2 \cdot a \cdot 2 \cdot a \cdot 2 \cdot a \cdot 2 \cdot a \cdot 2 \cdot a$

DP link

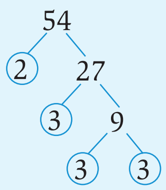
This investigation is similar to the style of investigation which you will encounter if you study the MAA course. You will look at a variety of specific examples and then try to use these to suggest a generalization or rule.

Higher Level

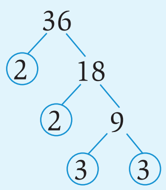
1.7 Greatest common factor and least common multiple

Example 7

Find the greatest common factor of 54 and 36.



$54 = 2 \times 3 \times 3 \times 3$



$36 = 2 \times 2 \times 3 \times 3$

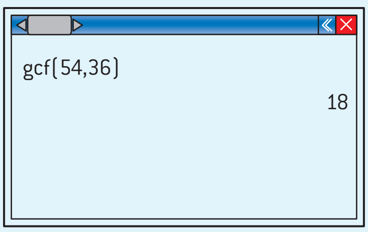
Begin dividing each number by the smallest prime number which is a factor. Here, divide by each of the prime numbers 2, 3, … in turn until you reach an answer 1.

Write each number as a **product** of the divisors.

$2 \times 3 \times 3 = 18$

The greatest common factor of 54 and 36 is 18.

Find the product of all the factors that are common to both numbers.



Most GDCs have a function that will find the greatest common factor.

Key point

The **greatest common factor** (divisor) of two (or more) numbers is the largest number that will divide into them both.

Note

A **product** is two or more numbers multiplied together.

Key point

The **least common multiple** of two (or more) numbers is the smallest number that both (or all) the numbers will divide into.

DP link

In Maths DP, HL students also learn to generalize results of greatest common factor and least common multiple in algebra and use them for factorization and combining algebraic fractions.

Example 8

Find the least common multiple of 15 and 25.

The multiples of 15 are:

15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, 210, 225, 240, …

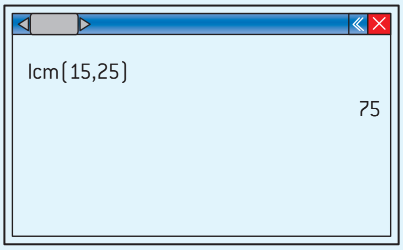
The multiples of 25 are:

25, 50, 75, 100, 125, 150, 175, 200, 225, 250, …

List multiples of each number

The least common multiple of 15 and 25 is 75.

Find the smallest number that is in each list.



Most GDCs have a function that will find the least common multiple.

Exercise 1.7

1 Find the greatest common factor of

a 36 and 20 **b** 6 and 12 **c** 18 and 42 **d** 36, 54 and 90.

2 Find the least common multiple of

a 6 and 12 **b** 8 and 20 **c** 15 and 40 **d** 4, 6 and 21.

DP style **Applications and Interpretation HL**

3 The size of populations of different species often follows a periodic cycle. Often the population will increase to a certain level and then decrease due to competition. The length of the cycle will be the time between successive maximums.

Suppose three species have cycles of length 4, 6 and 9 years respectively, and suppose all populations are at their maximum at year 0.

a Find when the populations will again all be at their maximum together.

Periodical cicadas emerge in a swarm after a fixed number of years. Assume the cicadas are eaten by all three of the species above.

b If a population of cicadas emerged in year 0 and periodically every 12 years after that, how many of the next ten emergences would match with **i** one **ii** two **iii** three maximums of the predator populations.

In fact, the number of years between emergences for populations of periodic cicadas are almost always prime numbers.

c Repeat part **b** given that the population emerges every 13 years.

d Two species of periodic cicadas share the same territory. The periods of the two groups are 13 and 17 years. Explain why these numbers will provide an evolutionary advantage over the similar length cycles of 12 and 18 years.

DP style | Analysis and Approaches HL

4 a i Write 30 and 135 as products of their prime factors.

ii By consideration of the factors, show that the product of 30 and 135 is equal to the product of their highest common factor and least common multiple.

Consider two numbers $m, n \in \mathbb{Z}$. Let h be the highest common factor of m and n and l be the least common multiple of m and n.

b Show that $m \times n = h \times l$

1.8 Fractions

Although you can perform these numerical calculations with a calculator, your GDC cannot perform operations with algebraic fractions in the same way. It is therefore important that you understand how to perform these operations without your GDC so you know the methods required if you start calculating with algebraic fractions.

The way you write a fraction is not unique. For every fraction there are **equivalent fractions**. Equivalent fractions have the same value but have different denominators, such as $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$. The fraction you should use is one that is in its **simplest terms**, that is, a fraction that you have cancelled as far as you can.

DP link

In Maths DP, HL students also learn to manipulate algebraic fractions.

Example 9

a Find fractions that are equivalent to $\frac{2}{3}$.

b Write $\frac{60}{108}$ in its simplest terms.

a $\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$

Multiply the numerator and denominator by the same number to find an equivalent fraction.

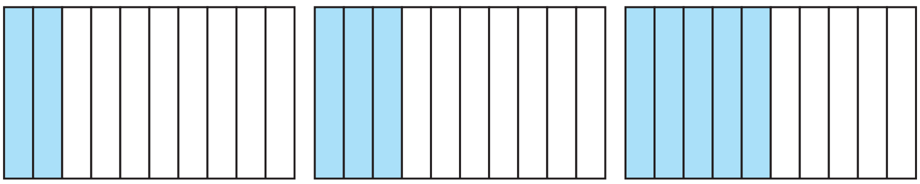
b $\frac{60}{108} = \frac{60 \div 2}{108 \div 2} = \frac{30}{54} = \frac{30 \div 2}{54 \div 2} = \frac{15}{27} = \frac{15 \div 3}{27 \div 3} = \frac{5}{9}$

write this as $\frac{\cancel{60}}{\cancel{108}} = \frac{\cancel{30}}{\cancel{54}} = \frac{\cancel{15}}{\cancel{27}} = \frac{5}{9}$

Divide by common factors until no more dividing is possible. This is called 'cancelling'.

When you add (or subtract) fractions, they must have the same denominator. You should write any answer in its simplest terms.

For example: $\frac{2}{10} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}$



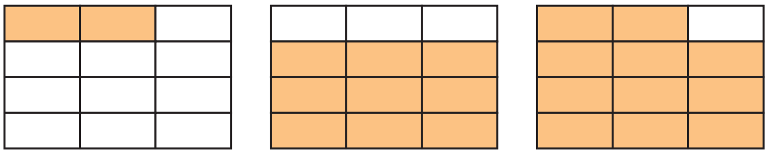
$\frac{2}{10}$ $+$ $\frac{3}{10}$ $\frac{5}{10} = \frac{1}{2}$

DP ready **International-mindedness**

The Egyptians wrote fractions as a sum of fractions which all had a numerator of 1. For example, you can write $\frac{2}{3}$ as $\frac{1}{3}+\frac{1}{4}+\frac{1}{12}$. The Babylonians, who lived in present-day Iraq, used a sexagesimal system based on 60. They had a method similar to our decimal system. $0.11=\frac{1}{10}+\frac{1}{100}$ is equivalent to $\frac{6}{60}+\frac{36}{60^2}$. The present-day system of writing fractions was not introduced in Europe until the 17th century.

If you are adding fractions that do not have the same denominator, you must first find equivalent fractions with a **common denominator**.

For example, $\frac{1}{6}+\frac{3}{4}=\frac{2}{12}+\frac{9}{12}=\frac{11}{12}$.



$\frac{1}{6}\left(\frac{2}{12}\right) \quad + \quad \frac{3}{4}\left(\frac{9}{12}\right) \quad = \quad \frac{11}{12}$

Multiplication of fractions is the process of finding one fraction of another. For example, to find $\frac{3}{4}\times\frac{5}{8}$ you have to find $\frac{3}{4}$ of $\frac{5}{8}$.

$\frac{3}{4} \quad \times \quad \frac{5}{8} \quad = \quad \frac{15}{32}$

You can see in the diagram above that you multiply fractions by multiplying the numerators together and multiplying the denominators together.

For example, $\frac{3}{4}\times\frac{5}{8}=\frac{3\times 5}{4\times 8}=\frac{15}{32}$

Because numerators are multiplied, and denominators are multiplied, you can cancel fractions 'diagonally' before multiplication in order to make the calculation simpler.

Find $\frac{1}{6}\times\frac{4}{5}$.

Dividing top and bottom by 2 gives $\frac{1}{\cancel{6}_3}\times\frac{\cancel{4}^2}{5}=\frac{1\times 2}{3\times 5}=\frac{2}{15}$.

Dividing by 2 is the same as multiplying by $\frac{1}{2}$ and dividing by 4 is the same as multiplying by $\frac{1}{4}$. It follows that dividing by $\frac{1}{2}$ is the same as multiplying by 2 and dividing by $\frac{1}{4}$ is the same as multiplying by 4. $\frac{1}{2}$ is called the **reciprocal** of 2 and $\frac{1}{4}$ is the reciprocal of 4.

What happens if you divide by $\frac{3}{4}$?

First, split this into dividing by 3 and dividing by $\frac{1}{4}$. This is the same as multiplying by $\frac{1}{3}$ and multiplying by 4. Putting these back together, this is the same as multiplying by $\frac{4}{3}$; the reciprocal of $\frac{3}{4}$.

For example, $\frac{7}{8}\div\frac{3}{4}=\frac{7}{\cancel{8}_2}\times\frac{\cancel{4}^1}{3}=\frac{7\times 1}{2\times 3}=\frac{7}{6}$.

Key point

Division is equivalent to multiplication by a reciprocal.

The answer to this example is what is known as an **improper fraction** as the numerator is greater than the denominator. You can write the answer as a **mixed number**, $\frac{7}{6} = 1\frac{1}{6}$.

Exercise 1.8

1 Write in simplest terms

a $\frac{12}{28}$ **b** $\frac{15}{40}$ **c** $\frac{18}{54}$ **d** $\frac{125}{1000}$

2 Calculate

a $\frac{3}{11} + \frac{5}{11}$ **b** $\frac{1}{3} + \frac{1}{6}$ **c** $\frac{3}{5} - \frac{1}{4}$ **d** $\frac{3}{8} + \frac{5}{12}$

3 Calculate

a $\frac{1}{2} \times \frac{3}{5}$ **b** $\frac{2}{3} \times \frac{9}{16}$ **c** $\frac{3}{4} \div \frac{5}{8}$ **d** $\frac{1}{2} \div \frac{1}{8}$

1.9 Exponential expressions

An **exponent** (or **index**) is a **power** of a number.

3 is the exponent

$$8^3 = \underbrace{8 \times 8 \times 8}_{3 \text{ times}}.$$

The exponent is the number of times you multiply the number by itself.

You can write:

$2 \times 2 \times 2 \times 2$ as 2^4

3×3 as 3^2

$10 \times 10 \times 10 \times 10 \times 10 \times 10$ as 10^6.

You can also write 2 as 2^1 or 4 as 4^1.

Key point

$a^1 = a$

$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$

Investigation 1.3

1 $2^1 \times 2^2 = (2) \times (2 \times 2) = 2 \times 2 \times 2 = 2^3$

Multiply $2^2 \times 2^3$, $2^1 \times 2^4$, $2^2 \times 2^2$, $2^3 \times 2^4$ writing your answers as powers of 2.

Do you notice a pattern? Can you generalize and find a rule for combining the powers of 2 when you multiply?

Multiply $3^2 \times 3^3$ and $5^1 \times 5^2$.

Does your rule apply to these multiplications as well?

2 $2^5 \div 2^2 = \frac{2 \times 2 \times 2 \times \cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2}} = 2 \times 2 \times 2 = 2^3$

Divide $2^4 \div 2^3$, $2^6 \div 2^4$, $2^3 \div 2^2$, $2^5 \div 2^1$ writing your answers as powers of 2.

Is there a pattern? Can you find a rule for division with powers of 2?

Try dividing some powers of 3 and 5. Does this rule apply to division with powers of these numbers too?

Key point

You can combine indices according to these rules:

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{m\times n}$$
$$a^n \times b^n = (a \times b)^n$$

DP link

In DP Maths, HL and SL students learn about logarithms, which rely on these laws of indices.

3 $(2^2)^3 = (2\times 2)\times(2\times 2)\times(2\times 2) = 2\times 2\times 2\times 2\times 2\times 2 = 2^6$

Calculate these powers of powers: $(2^1)^4$, $(2^2)^2$, $(2^3)^2$, writing your answers as powers of 2.

Is there a pattern? Can you generalize and find a rule for powers of powers of 2?

Try this with powers of other numbers. Does the rule you found apply to these numbers too?

4
$$\begin{aligned} 2^3 \times 3^3 &= (2\times 2\times 2)\times(3\times 3\times 3) \\ &= 2\times 2\times 2\times 3\times 3\times 3 \\ &= 2\times 3\times 2\times 3\times 2\times 3 \\ &= (2\times 3)\times(2\times 3)\times(2\times 3) \\ &= 6\times 6\times 6 \\ &= 6^3 \end{aligned}$$

Calculate $2^2 \times 4^2$, $3^2 \times 4^2$, $2^4 \times 5^4$

Is there a pattern? Can you generalize and find a rule for multiplying the same power of two numbers?

Example 10

Find the value of each expression. Where possible, use the laws of indices to first simplify the expression.

a $2^3 \times 2^4$ **b** $3^2 \times 4^2$ **c** $2^2 \times 3^2 \times 2^3 \times 3^3 \times 5^2$ **d** $6^5 \div 6^3$

e $\dfrac{2^4 \times 3^2 \times 3^3}{2^2 \times 3^4}$ **f** $(3^2)^2$ **g** $\sqrt{225}$ **h** $\sqrt[3]{27}$

a $2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$	Add indices when multiplying.
b $3^2 \times 4^2 = (3\times 4)^2 = 12^2 = 144$	Same index.
c $2^2 \times 3^2 \times 2^3 \times 3^3 \times 5^2 = 2^{2+3} \times 3^{2+3} \times 5^2 = 2^5 \times 3^5 \times 5^2 = (2\times 3)^5 \times 25 = 6^5 \times 25 = 7776 \times 25 = 194\,400$	Deal with powers of different numbers separately.
d $6^5 \div 6^3 = 6^{5-3} = 6^2 = 36$	Subtract indices when dividing.
e $\dfrac{2^4 \times 3^2 \times 3^3}{2^2 \times 3^4} = 2^{4-2} \times 3^{2+3-4} = 2^2 \times 3^1 = 4 \times 3 = 12$	A fraction line implies division.
f $(3^2)^2 = 3^{2\times 2} = 3^4 = 81$	Multiply the indices.
g $\sqrt{225} = 15$	$\sqrt{225}$ is the number which when squared gives 225. The value can be found using the GDC.
h $\sqrt[3]{27} = 3$	$\sqrt[3]{27}$ is the number which when cubed gives 27. Again, the value can be found using the GDC.

Exercise 1.9a

1 Find:

a 2^5 **b** 3^3 **c** 10^6

d $(-3)^5$ **e** -4^2

2 Calculate:

a $5^2 \times 5^2$ **b** $4^6 \div 4^4$ **c** $2^3 \times 5^2 \times 2^2 \times 5^3$

d $\left(2^3\right)^4$ **e** $\dfrac{3^2 \times 3^4}{3^3}$ **f** $\dfrac{2^2 \times 3^3}{2 \times 3}$

g $\dfrac{4^3 \times 5^4}{4^4 \times 2 \times 5^2}$ **h** $\left(7^2\right)^3 \div 7^4$

3 Use a calculator to find:

a 1.4^6 **b** 2.53^5 **c** 1.025^{14}

d $(-0.3)^5$ **e** $(-2.5)^4$

DP link

DP students will study the laws of exponents in greater detail in both MAA and MAI at SL and HL

Higher Level

So far in section **1.9**, you have only dealt with indices that are positive integers.

Look at 2^0. Applying the rules of indices, $2^2 \div 2^2 = 2^{2-2} = 2^0$. **Compare** this to $4 \div 4 = 1$. It follows that $2^0 = 1$.

What does 2^{-1} mean? Applying the rules of indices,

$$\begin{aligned} 2^1 \div 2^2 &= 2^{1-2}. \\ &= 2^{-1} \end{aligned}$$

Compare this to $2 \div 4 = \frac{1}{2}$.

You can **deduce** that $2^{-1} = \frac{1}{2}$.

Now look at $2^{\frac{1}{2}}$. What does this mean? Applying the rules of indices,

$$\begin{aligned} 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} &= 2^{\frac{1}{2}+\frac{1}{2}} \\ &= 2^1 \\ &= 2 \end{aligned}$$

Compare this to $\sqrt{2} \times \sqrt{2} = 2$.

You can deduce that $2^{\frac{1}{2}} = \sqrt{2}$.

Similarly $2^{\frac{1}{3}} = \sqrt[3]{2}$

Generalizing from these results:

Command term

Compare means you should give an account of the similarities between two items, referring to both of them throughout.

Deduce means you should reach a conclusion from the information given.

Internal link

You use this result to express numbers in standard form.

Key point

$a^0 = 1$

$a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{1}{n}} = \sqrt[n]{a}$

Example 11

Find:

a 10^{-3} **b** $4^{-2} \times 4^3$ **c** $3^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ **d** $\dfrac{2^3 \times 2^{-4}}{2^{-1}}$

a $10^{-3} = \dfrac{1}{10^3} = \dfrac{1}{1000} = 0.001$

b $4^{-2} \times 4^3 = 4^{-2+3} = 4^1 = 4$

c $3^{\frac{1}{2}} \times 2^{\frac{1}{2}} = (3 \times 2)^{\frac{1}{2}} = \sqrt{6}$	Both numbers are to the same power.
d $\frac{2^3 \times 2^{-4}}{2^{-1}} = 2^{3-4-(-1)} = 2^0 = 1$	Subtract the negative index to divide.

Look at $2^{\frac{3}{2}}$. Since you can write $\frac{3}{2} = \frac{1}{2} \times 3$

$2^{\frac{3}{2}} = \left(2^{\frac{1}{2}}\right)^3$ or $2^{\frac{3}{2}} = (2^3)^{\frac{1}{2}}$

$= \left(\sqrt{2}\right)^3$ $\quad = \sqrt{2^3}$

Generalizing from this result we get the key point here.

Key point

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$$

$$= \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

Example 12

Find $8^{\frac{4}{3}}$

$8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16$	Notice that it is easier to calculate $\left(\sqrt[3]{8}\right)^4$ than $\sqrt[3]{8^4} = \sqrt[3]{4096}$

Exercise 1.9b

1 Calculate:

a 3^{-2} **b** 5^{-1} **c** 4^{-3} **d** $\left(\frac{1}{2}\right)^{-1}$ **e** $\left(\frac{2}{3}\right)^{-2}$

2 Find:

a $5^{-2} \times 2^4$ **b** $3^{-1} \times 3$ **c** $12 \times 3^{-1} \times 2^{-2}$ **d** $\frac{5^2}{5^4}$ **e** $\frac{2^3 \times 3^2}{2^5 \times 3^3}$

3 Calculate:

a $27^{\frac{1}{3}}$ **b** $\left(\frac{16}{9}\right)^{\frac{1}{2}}$ **c** $4^{\frac{5}{2}}$ **d** $32^{\frac{3}{5}}$ **e** $9^{-\frac{1}{2}}$

DP style Analysis and Approaches HL

4 **a** **i** Find the greatest common factor of 90 and 135.

ii Hence prove $3^{135} > 5^{90}$

b Find a counter example to show that the statement $\left(a^n\right)^{\frac{1}{n}} = a$ is not always true.

DP style Applications and Interpretation HL

5 Kepler's third law of planetary motion states that the average distance of a planet from the sun (R), where R is measured in astronomical units (AU), is related to the period (T) of its orbit by the equation

$R^3 = kT^2$

where T is time in days, and k is a constant.

a Given that the average distance of the earth from the sun is one astronomical unit (AU) and that the earth takes 365.25 days to orbit the sun, find the value of k.

b The orbital period of Jupiter is 4333 days. Find its average distance from the sun in astronomical units.

c Rewrite the equation in the form $T = k_1 R^a$

d Hence find the orbital period of Neptune given its average distance from the sun is 30.07 AU.

1.10 Surds (radicals)

Numbers like $\sqrt{2}, \sqrt{3}$ and $\sqrt{7}$ are **surds** (or **radicals**). Surds are irrational numbers.

Key point

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ and $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

Note

Square numbers, such as $\sqrt{4}$, are not surds as, for example, $\sqrt{4} = 2$.

Investigation 1.4

Look at these expressions:

$$\sqrt{2} \times \sqrt{6} \qquad \frac{6}{\sqrt{3}} \qquad \frac{12}{\sqrt{12}} \qquad \frac{12}{2\sqrt{3}} \qquad \frac{12 - 6\sqrt{3}}{2 - \sqrt{3}} \qquad \sqrt{12}$$

$$\frac{6 + 2\sqrt{3}}{1 + \sqrt{3}} \qquad 2\sqrt{3} \qquad \frac{6\sqrt{2}}{\sqrt{6}} \qquad \frac{2\sqrt{6}}{\sqrt{2}} \qquad \frac{12}{\sqrt{2} \times \sqrt{6}}$$

1 Can you find any that are equal? If necessary, use your calculator to evaluate them. (Some GDCs will evaluate the expressions as decimals while others will express them in terms of surds unless you use the keypress that gives a decimal answer).

2 Which of the expressions that are equal do you think are the simplest?

3 Calculate $2\sqrt{3}$ to 3 s.f. and square your answer. Calculate $\left(2\sqrt{3}\right)^2$. Compare your answers. Which is the most accurate?

You should write expressions containing surds so that you cannot simplify them any further.

DP link

This style of investigation is similar to those contained in both the MAI and MAA courses.

Example 14

Simplify:

a $\sqrt{18}$ **b** $\sqrt{6} \times \sqrt{8}$

a $\sqrt{18} = \sqrt{2 \times 9} = \sqrt{2} \times \sqrt{9} = \sqrt{2} \times 3 = 3\sqrt{2}$	Look for factors that are square numbers
b $\sqrt{6} \times \sqrt{8} = \sqrt{6 \times 8} = \sqrt{48} =$ $= \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$	Combine terms and then look for square factors

Exercise 1.10a

1 Simplify:

a $\sqrt{32}$ **b** $\sqrt{12} \times \sqrt{10}$

2 Simplify:

a $\sqrt{2}\left(3 + \sqrt{2}\right)$ **b** $3\sqrt{2} + \sqrt{8} - \sqrt{18}$

DP link

DP students taking MAA at SL and HL will study exact values of trigonometric ratios using surds.

Higher Level

Rationalizing the denominator

An expression is simpler if it has a rational denominator.

To rationalize the denominator in an expression like $\dfrac{1}{\sqrt{2}}$ you multiply the numerator and denominator by $\sqrt{2}$.

You can also simplify more complicated expressions like $\dfrac{1}{1 + \sqrt{2}}$ by multiplying top and bottom by the equivalent expression with opposite sign in the middle, e.g. $1 - \sqrt{2}$

Example 14

Simplify **a** $\frac{1}{\sqrt{2}}$ **b** $\frac{6}{5\sqrt{3}}$ **c** $\frac{1}{1+\sqrt{2}}$

a $\frac{1}{\sqrt{2}} = \frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{\sqrt{2}}{2}$

Multiply numerator and denominator by $\sqrt{2}$

b $\frac{6}{5\sqrt{3}} = \frac{6\times\sqrt{3}}{5\sqrt{3}\times\sqrt{3}} = \frac{6\sqrt{3}}{15} = \frac{2\sqrt{3}}{5}$

Multiply numerator and denominator by $\sqrt{3}$

c $\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$

Multiply numerator and denominator by $1-\sqrt{2}$. If $1+\sqrt{2}$ is $a+b$ then $1-\sqrt{2}$ is $a-b$.

$= \frac{1-\sqrt{2}}{1^2-(\sqrt{2})^2}$

Use the difference of 2 squares result (see Internal link box below)

$= \frac{1-\sqrt{2}}{1-2}$

simplify

$= \frac{1-\sqrt{2}}{-1}$

divide by −1

$= \sqrt{2}-1$

Note

The rationalized form is simpler because its relative size can easily be seen and it can then be used to add surds.

Internal link

Multiplying top and bottom by the equivalent expression with opposite sign in the middle comes from a result called the difference of two squares: $(a+b)(a-b) = a^2 - b^2$. You will learn about this in chapter 2.

Key point

To rationalize the denominator when it is $\sqrt{a} \pm \sqrt{b}$ you multiply the numerator and denominator of the fraction by $\sqrt{a} \mp \sqrt{b}$.

Exercise 1.10b

1 Simplify

a $\frac{2}{\sqrt{6}}$ **b** $\frac{\sqrt{24}}{2\sqrt{3}}$ **c** $\frac{\sqrt{3}\times\sqrt{10}}{\sqrt{12}\times\sqrt{5}}$

2 Simplify

a $\frac{1}{2+\sqrt{3}}$ **b** $\frac{9}{\sqrt{5}-\sqrt{2}}$ **c** $\frac{6}{3\sqrt{2}+2}$

d $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ **e** $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{8}+2\sqrt{3}}$

1.11 Standard form

In science, you often have to deal with very large and very small numbers.

To help make large and small numbers more comprehensible, you can use **standard form** (or **scientific notation**).

Did you know

For example, the average distance from the Earth to the Sun is about 149 598 000 000 m, one atomic weight unit is 0.000 000 000 000 000 000 000 000 001 66 kg and the surface area of the earth is 453 000 000 000 000 m^2.

Key point

In standard form you write numbers in the form $a \times 10^n$ where $1 \le a < 10$ and $n \in \mathbb{Z}$.

To be able to use standard form, you need to use powers of 10. Remember that $10^1 = 10$, $10^3 = 1000$, etc. You will also need to know that $10^0 = 1$ and that $10^{-n} = \frac{1}{10^n}$, for example $10^{-3} = \frac{1}{10^3} = 0.001$.

Numbers written in standard form are easier to compare and easier to calculate with.

Internal link

Section 1.9 explained indices or powers.

DP link

In DP HL students will learn more about indices that are not positive integers.

DP link

DP students will study operations with numbers in standard form in MAA and MAI at SL and HL.

Example 15

1 Write these numbers in standard form.

a 149 598 000 000 **b** 0.000 000 000 000 000 000 000 000 001 66

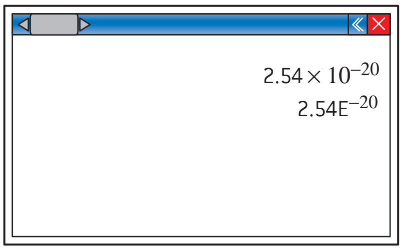
c 453 000 000 000 000

a $149\,598\,000\,000 = 1.496 \times 100\,000\,000\,000$
$= 1.496 \times 10^{11}$

b $0.000\,000\,000\,000\,000\,000\,000\,000\,001\,66$
$= 1.66 \times 0.000\,000\,000\,000\,000\,000\,000\,000\,001$
$= 1.66 \times 10^{-27}$

c $453\,000\,000\,000\,000$
$= 4.53 \times 100\,000\,000\,000\,000$
$= 4.53 \times 10^{14}$

Your calculator will express answers that are very large or very small in standard form. Some calculators use a recognizable index notation, but others use the symbol E. This form of calculator notation (seen in some computer applications too) is **not** acceptable in the DP course. For example, you should write 3.4×10^3 and **not** 3.4E3.



You cannot always give an answer in standard form, so you will need to know how to write it as a decimal number.

Example 16

Write these numbers as decimal numbers.

a 2.12×10^2 **b** 3.58×10^4 **c** 8.05×10^{-1} **d** 6.95×10^{-5}

a $2.12 \times 10^2 = 212$	Move the point two places to the right.
b $3.58 \times 10^4 = 35\,800$	Move the point four places to the right.
c $8.05 \times 10^{-1} = 0.805$	Move the decimal point one place to the left.
d $6.95 \times 10^{-5} = 0.0000695$	Move the point 5 places to the left and fill extra places with zeros.

The way you enter numbers in standard form also depends on the GDC you have. To enter the number 2.54×10^{-20} you should type either 2.54 × [10^x] –20 or 2.54 [E] –20, depending on which GDC you are using. You could use the 10^x key and type 2.54 × 10^x –20 or you could use the ^ key and type 2.54 × 10 ^ –20, but the standard form key E requires only one keypress and is easier to use.

Exercise 1.11

1 Jupiter's diameter is 1.43×10^5 km and its mean distance from the Sun is 7.78×10^8 km.
Saturn's diameter is 1.21×10^5 km and its mean distance from the Sun is 1.43×10^9 km.
State which of the two planets is farthest from the Sun and which is the largest.

2 Write these numbers in standard form.

a 324 000 000 b 456 000 c 0.000 128 d 0.000 006 21

3 Write these numbers as decimal numbers.

a 2.50×10^3 b 4.81×10^1 c 2.85×10^{-2} d 3.07×10^{-4}

DP style **Applications and Interpretation SL**

4 Protons and neutrons have a mass of 1.67×10^{-27} kg and the mass of an electron is 9.11×10^{-31} kg.

a Calculate how many times more massive a proton or neutron is than an electron.

b An oxygen atom has 8 protons, 8 neutrons and 8 electrons. Calculate how much it weighs.

Chapter summary

- ☐ Numbers:
  - ☐ The natural numbers, $\mathbb{N}$, are the counting numbers (including the number 0): 0, 1, 2, 3, 4, 5, …
  - ☐ The integers, $\mathbb{Z}$, include all the natural numbers and their negative values too: … −2, −1, 0, 1, 2, …
  - ☐ The rational numbers, $\mathbb{Q}$, include all the integers and numbers in the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers. Examples of rational numbers are $\frac{5}{6}, -\frac{1}{2}, \frac{20}{7}, 3\frac{1}{2}, \ldots$, terminating decimals like 0.5, 1.75, −7.396, and recurring decimals like 0.333… and 1.7272… .
  - ☐ The real numbers, $\mathbb{R}$, contain both the rational and irrational numbers. Irrational numbers include $\pi, \sqrt{2}, \sqrt[3]{10}$ and any non-terminating, non-recurring decimals.
- ☐ Sets:
  - ☐ A **set** is a collection of objects, for example a collection of numbers, letters, geometrical objects or anything else.
  - ☐ The number of elements in a set A is its **cardinality**, $n(A)$.
  - ☐ The set that contains no elements at all is the **empty set**, $\varnothing$. $n(\varnothing) = 0$.
  - ☐ The set of all the elements you are considering is the **universal set**, U.
  - ☐ The set of elements of U that are not in A is its **complement**, A'. $A' = \{x \mid x \in U, x \in A\}$.
  - ☐ The set of elements that are either in A or B or both is the **union** of A and B, $A \cup B$.
  - ☐ The set of elements that are in both A and B is the **intersection** of A and B, $A \cap B$.
- ☐ When rounding, consider the figure immediately to the right of the last digit you are rounding to. If the next figure is 0, 1, 2, 3, or 4, round down. If the next figure is 5, 6, 7, 8, or 9, round up.
- ☐ The **modulus** or absolute value $|x|$ of x is $\begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$.
- ☐ You can summarize the rules for the order of operations as:
  1. **B**rackets (or parentheses),
  2. **O**rders (indices or exponents),
  3. **M**ultiplication/**D**ivision,
  4. **A**ddition/**S**ubtraction,

You can remember this using the mnemonic **BOMDAS**

- ☐ A **prime number** is a positive integer, greater than 1 that has exactly two factors. It is not a multiple of any other number apart from 1 and itself. Numbers that have more than two factors are composite. The number 1 is neither prime nor composite.
- ☐ The **greatest common factor** (divisor) of two (or more) numbers is the largest number that will divide into them both.

- ☐ The **least common multiple** of two (or more) numbers is the smallest number that the numbers will divide into.
- ☐ Fractions:
  - ☐ Equivalent fractions have the same value
  - ☐ If you are adding fractions that do not have the same denominator, you must first find equivalent fractions in order to write them with a **common denominator**
  - ☐ When multiplying fractions, multiply the numerators and multiply the denominators
  - ☐ Division is equivalent to multiplication by a reciprocal
- ☐ Rules of indices:
  - ☐ $a^1 = a$

    $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$
  - ☐ $a^m \times a^n = a^{m+n}$

    $a^m \div a^n = a^{m-n}$

    $(a^m)^n = a^{m \times n}$

    $a^n \times b^n = (a \times b)^n$
- ☐ $a^0 = 1$ (HL)
- ☐ $a^{-n} = \frac{1}{a^n}$ (HL)
- ☐ $a^{\frac{1}{n}} = \sqrt[n]{a}$ (HL)
- ☐ $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$ (HL)

  $= (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$
- ☐ $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- ☐ To rationalize the denominator when it is $\sqrt{a} \pm \sqrt{b}$ you multiply the numerator and denominator of the fraction by $\sqrt{a} \mp \sqrt{b}$.
- ☐ In standard form you write numbers in the form $a \times 10^n$ where $1 \le a < 10$ and $n \in \mathbb{Z}$.

Chapter 1 test

DP style Analysis and Approaches SL

1 $P = \{p \mid p \in \mathbb{Z}, 4 \le p < 10\}$ and $Q = \{q \mid q \in \mathbb{N}, q \le 7\}$, where $U = \mathbb{Z}$

- **a** Write the sets P and Q as lists inside curly brackets { }.
- **b** Determine which of these statements are true and which are false:

  i $8 \in P \cap Q$ **ii** $-2 \in P \cap Q$ **iii** $10 \in P'$ **iv** $0 \in P \cup Q$
- **c** Write the set $P \cap Q$ using set builder notation.

2 Calculate 2.84×37.6, giving your answer to:

- **a** 3 s.f.
- **b** 2 d.p.

3 Find

- **a** $|3 \times (-4)| - |3| \times |-4|$
- **b** $|3 - (-2)| - |3| - |-2|$
- **c** $|(-4)^3| + |-4|^3$

4 Determine which of these statements are true and which are false:

- **a** $2 + (4 - 5) = (2 + 4) - 5$
- **b** $(3 \times 4) \div 2 = 3 \times (4 \div 2)$
- **c** $6 - (2 - 3) = (6 - 2) - 3$
- **d** $24 \div (6 \div 2) = (24 \div 6) \div 2$

5 State which of these numbers is prime. If a number is composite, write it as a product of prime factors.

a 57 **b** 73 **c** 97 **d** 143 **e** 133

6 Simplify each of the following expressions:

a $\frac{3}{4}-\frac{1}{8}$

b $\frac{1}{2}\times\frac{5}{8}$

c $\frac{10}{27}\div\frac{5}{12}$

d $\frac{2}{5}+\frac{1}{10}\times\frac{3}{4}$

7 Simplify each expression.

a $10^3\times10^2$ **b** $8^5\div8^3$ **c** $\frac{2^4\times10^2}{5^3}$

8 Simplify each of the following expressions:

a $\sqrt{18}$

b $\sqrt{24}\div\frac{\sqrt{2}}{2}$

c $\frac{\sqrt{2}}{2}\times\sqrt{24}\times\sqrt{3}$

9 a Write these numbers in standard form:

i 123 580 000 **ii** 0.00127

b Write these numbers in decimal form:

i 2.54×10^5 **ii** 7.68×10^{-2}

DP style Analysis and Approaches SL

10 All numbers in this question are written in standard form.

a Given that

$$\frac{1.5\times10^p}{2\times10^q}=a\times10^r$$

i find the value of a

ii find an expression for r in terms of p and q

b i find an expression for d in terms of b and c given that $b\times10^6+c\times10^7=d\times10^7$

ii state an additional constraint that must be satisfied by b and c and justify your answer.

DP style Applications and Interpretation SL

11 Under certain conditions the size of a population of fruit flies can be modelled by the equation

$$N=a2^{bt}$$

where N is the size of the population and t is the time in weeks from a fixed point.

A population of fruit flies in a large container initially (at $t=0$) has just 5 fruit flies. After 2 weeks there are 320 fruit flies.

Assuming the equation is a good model for the population:

a find the values of a and b

b find the size of the population after 3 weeks.

The equation is rewritten in the form $N=ac^t$.

c Write down the value of c.

d Find the first week on which the population exceeds 1000.

Higher Level

12 a Find the greatest common factor of 42 and 28.

b Find the least common multiple of 42 and 28.

13 Calculate **a** $2^{-2}\times2^6\times2^{-1}$ **b** $8^{\frac{1}{2}}\times8^{\frac{1}{6}}$ **c** $\frac{5^4\times5^{-2}}{5^2}$

14 Simplify **a** $\frac{6}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2}-1}$ **c** $\frac{\sqrt{3}}{\sqrt{3}+\sqrt{2}}$

Reflect

Did you prefer the MAA or MAI style questions? What do you need more practice in.

Modelling and investigation

DP ready **Approaches to learning**

Critical thinking: Analysing and evaluating issues and ideas

Communication: Reading, writing and using language to gather and communicate information

Self-management: Managing time and tasks effectively

DP link

This activity is similar in style to the MAI course as it investigates a real-life context.

Building a model of the solar system

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Diameter (km)	4879	12 104	12 756	6792	142 984	120 536	51 118	49 528
Distance from the Sun (10^6 km)	57.9	108.2	149.6	227.9	778.6	1433.5	2872.5	4495.1

Elizaveta is going to create a model of the solar system using some balls to represent planets. She has four types of ball available. In order of size with their approximate diameters, these are the balls:

tennis ball: 7 cm volleyball: 20 cm football: 22 cm basketball: 24 cm

She organizes the sizes of the planets into order and groups those that are a similar size.

- Mercury and Mars are the smallest; she represents each of these using a tennis ball.
- Venus and Earth are next in size; she represents each of these using a volleyball.
- Uranus and Neptune are bigger; she represents each of these using a football.
- The largest are Jupiter and Saturn; she represents each of these using a basketball.

To determine whether this arrangement forms an accurate scale model of the true planet sizes, Elizaveta divides the planet diameter (in km) by the diameter of the ball (in cm) used to represent it.

By considering these calculations, comment on whether Elizaveta be able to make a realistic model with these balls.

Elizaveta finds some Styrofoam balls with these diameters: 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 25, 30, 40 (measurements in cm).

Since Mercury is the smallest planet, she chooses the 1 cm ball to represent it.

How many times larger is Venus than Mercury, to 1 dp? Which ball would represent Venus?

How many times larger is Earth than Mercury, to 1 dp? Which ball would represent Earth?

Choose the balls whose sizes would be closest in scale to the other planets.

The distances from the Sun are in units of 10^6 km. If Elizaveta used the same scale to represent these as she did to represent the diameters, Neptune would be 9 km from the Sun in her model! Instead, she uses a scale that is much smaller so that her model will fit in the room.

She begins by placing Mercury 10 cm from the Sun.

How many times further away than Mercury is Venus? Calculate your answer to the nearest cm. How far away from the Sun will Venus be in the model?

On this scale how far would you place Neptune from the Sun? Elizaveta's classroom is 10 m by 10 m. Does she have room for her model?

What would the distances from the Sun be for the 8 planets?