OXFORD IB PREPARED

IN COOPERATION WITH ib

MATHEMATICS: ANALYSIS AND APPROACHES

IB PREPARED

IB DIPLOMA PROGRAMME

OXFORD

1 NUMBER AND ALGEBRA

1.1 NUMBER REPRESENTATION, PROOF AND THE BINOMIAL THEOREM

You must know:

- ✔ the difference between an equation and an identity
- ✔ the binomial theorem.

You should be able to:

- ✔ calculate with and express numbers in scientific notation
- ✔ construct simple deductive proofs
- ✔ apply the binomial theorem.

Number representation

For very small and very large numbers it is convenient to represent them in the form

$$a \times 10^k$$

where a is a real number, $1 \le a < 10$, and k is an integer. This is called scientific notation and is achieved by "moving" the decimal point, for example $132\,000 = 1.32 \times 10^5$ and $0.000000456 = 4.56 \times 10^{-7}$

Note

Using scientific notation will alter the number of decimal places, but it will not affect the number of significant figures. This is why writing numbers in scientific notation provides a better indicator of the level of accuracy involved. The second example above confirms that leading zeros are not significant.

Note

If the question asks you to give your answer exactly, you may write, for example: 134, π, $\sqrt{2}$, $\frac{13}{7}$ and so on. Do not give a rounded decimal as your answer if the question asks for an exact answer.

Assessment tip

The following statement will appear on the first page of all IB Mathematics papers.

Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.

Remember this when answering any question. Many students lose marks by not following this instruction. Remember that non-zero digits are significant, zeros between non-zeros are significant, but leading zeros are not significant.

Assessment tip

After any final numerical answer, you should state the accuracy that the answer has been given to, as shown by "(3 sf)" in Example 1.1.1. This is a check to yourself that you have followed the instructions in the question or at the start of the paper.

Example 1.1.1

Earth's moon can be modelled as a sphere with radius $r = 1740$ km. The formula for the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Use this formula to find the volume of the moon in km^3, giving your answer in the form $a \times 10^k$ where $a \in \mathbb{R}$, $1 \le a < 10$ and $k \in \mathbb{Z}$, i.e. in scientific notation.

Solution

$V = \frac{4}{3}\pi \times 1740^3 = 2.21 \times 10^{10}\,\text{km}^3 (3\text{ sf})$

Note

Calculators can use, for example, E6 to represent 10^6, where the E indicates an exponent. Remember that your answers should be given with correct notation and not using calculator nomenclature.

Simple deductive proof

An equation is true for some values of the variables, and so uses an equals sign "=".

An identity is true for all values of the variables, and so uses the symbol "≡".

For example, $3x + 1 = 7$ is an equation, whereas $(x + 1)^2 \equiv x^2 + 2x + 1$ is an identity.

An identity will have a right-hand side (RHS) and a left-hand side (LHS). To prove an identity, start with one side and use valid rules until that side has been transformed into the other side.

Example 1.1.2

Prove that $x^4 - 1 \equiv (x - 1)(x^3 + x^2 + x + 1)$, $x \in \mathbb{R}$

Solution

$$\begin{aligned} \text{RHS} &\equiv (x-1)(x^3 + x^2 + x + 1) \\ &\equiv x^4 + x^3 + x^2 + x - x^3 - x^2 - x - 1 \\ &\equiv x^4 - 1 \equiv \text{LHS} \end{aligned}$$

Assessment tip

It is often better to start with the side that looks the most complicated and try to simplify it to obtain the other side. Do not worry if the expression becomes longer before it eventually simplifies.

Note

You *cannot* prove an identity just by verifying that it is true for some value(s) of the variables. In Example 1.1.2, stating that both sides equal 0 for $x = 1$ does not prove that this identity is true for all $x \in \mathbb{R}$.

Assessment tip

The command terms used when a deductive proof is required are "prove" or "show that".

Note

You also *cannot* prove an identity by starting with the very statement you are trying to prove, working with it until you reach something that is true and then declaring the original to be true.

For example, the following statement is faulty logic as $A \Rightarrow B$ does not mean that $B \Rightarrow A$:

"$3 = 4 \Rightarrow 4 = 3 \Rightarrow$ by adding the equations together, $7 = 7$ and that is true, so $3 = 4$ must be true as well"

Sometimes both sides will seem complicated. It is then permissible to work with the LHS and show that this equals expression P, for example, and then start again independently with the RHS and show that this also equals P. Since both sides equal P, both sides are equal. However, you *cannot* start with both sides at the same time with an equals sign between them.

SAMPLE STUDENT ANSWER

Show that $x^2 + 6x + 13 \equiv (x+3)^2 + 4$

$x^2 + 6x + 13 = (x+3)^2 + 4$

$x^2 + 6x + 9 = (x+3)^2$

$x^2 + 6x + 9 = x^2 + 6x + 9$

$0 = 0$, so it is true

▲ The algebraic manipulation was useful and could have been part of a correct proof.

▼ The layout of the "proof" was incorrect. It started with the very statement that we were trying to prove and finished with $0 = 0$, which we already know. This working could be reconstructed into a proper proof as shown.

The answer above could have achieved 1/3 marks.

The correct solution should have been:

RHS $\equiv (x+3)^2 + 4 \equiv x^2 + 6x + 9 + 4 \equiv x^2 + 6x + 13 \equiv$ LHS

Binomial theorem

Note

The binomial theorem is given by

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + {}^nC_{n-1} a^1 b^{n-1} + b^n$

or in summation notation $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r}b^r$, where the ${}^nC_r$, the binomial coefficients (combinations or "choose" numbers), can be obtained from Pascal's triangle, the calculator or the formula ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

Assessment tip

In examinations, with "prove" or "show that" questions where the answer is given, your work will be checked carefully to ensure that each step follows logically from the previous step. Lay out your work methodically and do not miss out any steps. Remember, you are trying to communicate with the examiner. You are not just trying to convince yourself that the statement is true.

Assessment tip

The formula book will be available in all IB exams. Always work with it next to you. Get to know where each formula is within the book. Make sure that you have learned formulae that you will need that are not in the book.

Example 1.1.3

Expand $(1 + 2x)^4$ using the binomial theorem.

Solution

$(1 + 2x)^4 = 1^4 + 4 \times 1^3 \times (2x) + 6 \times 1^2 \times (2x)^2 + 4 \times 1^1 \times (2x)^3 + (2x)^4$

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

Assessment tip

When using the binomial theorem, identify what n, a and b are. For small values of n it is worth quickly writing down the start of Pascal's triangle:

1 1

1 2 1

1 3 3 1

1 4 6 4 1

For ${}^nC_r$, when you are looking at row n, remember that it starts with $r = 0$. Put the binomial coefficients spaced out on a line, then fill in the powers of a and b. It is worth putting brackets around the expressions that are a and b so that you realize that it is the whole bracket that is raised to the appropriate power.

The construction of Pascal's triangle relies upon the fact that

$${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$$

This formula can be proved using the formula for ${}^nC_r$ that involves factorials, but a more informal proof follows.

A set of r people is to be chosen from $n+1$ people. This can be done in ${}^{n+1}C_r$ different ways. The collection of $n+1$ people has one very special person called Colin. The sets of r people can be divided into two disjoint subsets: those that include Colin and those that do not. For sets that include Colin, we will have to choose Colin and will then have to choose $r-1$ people from the rest, so this can be done in ${}^nC_{r-1}$ different ways. For the sets that do not include Colin, all the r people will have to be chosen from the rest, so this can be done in ${}^nC_r$ different ways. Therefore, equating the two different ways of counting gives ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$.

SAMPLE STUDENT ANSWER

Expand $(2-3x)^3$ using the binomial theorem.

$2^3 + 3 \times 2^2 \times -3x + 3 \times 2 \times -3x^2 + 1 \times -3x^3$

$= 8 - 36x - 18x^2 - 3x^3$

The answer above could have achieved 2/4 marks.

The correct solution should have been:
$2^3 + 3 \times 2^2 \times (-3x) + 3 \times 2 \times (-3x)^2 + 1 \times (-3x)^3 = 8 - 36x - 54x^2 - 27x^3$

▲ The binomial theorem was used with the correct binomial coefficients and correct powers of a, ensuring that the first two terms were correct.

▼ Brackets were not placed around the expression for b which meant that powers of the -3 were ignored.

Example 1.1.4

Find the first three terms in the binomial expansion of $(1-4x)^{10}$ in ascending powers of x.

Solution

$(1-4x)^{10} = 1 + {}^{10}C_1\,(-4x) + {}^{10}C_2\,(-4x)^2 \ldots = 1 - 40x + 720x^2\ldots$

Assessment tip

For larger values of n, when evaluating ${}^nC_r$ use your calculator on the Calculator paper, and the formula involving factorials on the Non-calculator paper.

Example 1.1.5

Find the constant term in the binomial expansion of $\left(x + \frac{2}{x^3}\right)^8$

Solution

The general term is ${}^8C_r\,x^{8-r}\left(\frac{2}{x^3}\right)^r$

The power of x is $8 - r - 3r = 8 - 4r$

Require this to be 0, so $r = 2$ and the term is ${}^8C_2\,x^6\left(\frac{2}{x^3}\right)^2 = 112$

Assessment tip

With questions like Example 1.1.5, don't waste time writing down all the terms. Look at the general term and then fit it to what is required. The constant term is the one that does not involve x at all, because the exponent of x is zero.

1.4 ALGEBRA (HL)

You must know:

- ✔ the difference between combinations and permutations
- ✔ the formula for the extended binomial theorem
- ✔ what a counterexample is.

You should be able to:

- ✔ count the number of ways of arranging objects
- ✔ apply the extended binomial theorem
- ✔ decompose a rational function into partial fractions
- ✔ construct a formal proof by induction
- ✔ construct a proof by contradiction
- ✔ solve systems of linear equations.

Counting principles

The combination number ${}^nC_r$ represents the number of ways of choosing r objects from n objects. The permutation number ${}^nP_r$ represents the number of different ways of arranging r objects from n objects in order. These can both be found on a calculator.

Formulae for combinations and permutations

$${}^nC_r = \frac{n!}{r!(n-r)!} \qquad {}^nP_r = \frac{n!}{(n-r)!}$$

If one operation can be done in n ways and a second operation can be done in m ways then the total number of ways of doing both operations is $n \times m$. Think of walking down a road that divides into n paths and then each of these paths divides into m paths. There would be $n \times m$ different routes that you could take.

Example 1.4.1

Find the number of ways of selecting a committee of 3 females and 2 males from a set of 7 females and 6 males.	Solution We need to choose 3 females from 7 and 2 males from 6. ${}^7C_3 \times {}^6C_2 = 525$

Example 1.4.2

Find the number of ways of selecting 3 people to the positions of: chair-person, treasurer and deputy chair-person, from a committee of 5 people.	**Solution** ${}^5P_3 = 60$ The positions the persons would hold matters, so this is a permutation not a combination.

Assessment tip

Think of the C numbers as "choose" or "select" numbers, where the order does not matter.

Think of the P numbers as "put" or "arrange" numbers, where the order does matter.

Link to Probability SL 4.5

The difference between combinations and permutations can be illustrated as follows. If a team of 4 relay race runners is to be chosen from a group of 6 athletes, then this is a combination and there would be ${}^6C_4 = 15$ ways of selecting them.

If it is to be decided who carries the baton 1st, 2nd, 3rd, and 4th then this is a permutation and the number of different ways in which this could be done would be ${}^6P_4 = 360$. The ${}^nP_r$ numbers are always bigger than the ${}^nC_r$ numbers by a factor of $r!$

▲ Good counting method shown for dealing with "at least one", looking at all possibilities and then taking away those that cannot happen.

SAMPLE STUDENT ANSWER

A team of four children is to be chosen from six girls and five boys. Find the number of ways in which this can be done if the team must contain at least one boy and at least one girl.

${}^{11}C_4 = 330$

But we cannot have all boys or all girls so

$330 - {}^5C_4 - {}^6C_4 = 330 - 5 - 15 = 310$

This answer could have achieved 5/5 marks.

Extension of the binomial theorem

The binomial theorem gives a finite expansion for $(a + b)^n$, where $n \in \mathbb{Z}^+$. This can be extended to negative integers and fractions and an infinite expansion is obtained. We utilize $(a+b)^p = a^p\left(1+\frac{a}{b}\right)^p$, where $p \in \mathbb{Q}$ and the following expansion.

Extended binomial theorem

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \dots$$

where $p \in \mathbb{Q}$, provided that the RHS converges, so we need $|x| < 1$.

This expansion is not given in the formula book and so should be remembered.

Link to Binomial Theorem SL 1.9 and Maclaurin Expansions AHL 5.19

Example 1.4.3

(a) Find the first three terms in the extended binomial expansion for $(1+x)^{\frac{1}{2}}$, in ascending powers of x.

(b) Hence find a rational approximation for $\frac{\sqrt{3}}{2}$

Solution

(a) $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\times -\frac{1}{2}}{2}x^2 + \frac{\frac{1}{2}\times -\frac{1}{2}\times -\frac{3}{2}}{6}x^3 + \dots = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$

(b) Taking $x = -\frac{1}{4}$ $\quad \left(\frac{3}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2} \simeq 1 - \frac{1}{8} - \frac{1}{128} - \frac{1}{1024} = \frac{887}{1024}$

Example 1.4.4

Show that $(2+3x)^{-1} = \frac{1}{2} - \frac{3x}{4} + \frac{9x^3}{8} - \frac{27x^3}{16} + \dots$

Solution

$$(2+3x)^{-1} = 2^{-1}\left(1+\frac{3x}{2}\right)^{-1} = \frac{1}{2}\left(1 + -1\times\frac{3x}{2} + \frac{-1\times-2}{2!}\left(\frac{3x}{2}\right)^2 + \frac{-1\times-2\times-3}{3!}\left(\frac{3x}{2}\right)^3 + \dots\right)$$

$$= \frac{1}{2} - \frac{3x}{4} + \frac{9x^2}{8} - \frac{27x^3}{16} + \dots$$

Partial fractions

Just as fractions can be decomposed into simpler fractions, e.g. $\frac{7}{12}=\frac{1}{3}+\frac{1}{4}$, rational functions can be decomposed into simpler expressions. A simple deductive proof could be used to show that, for instance:

$$\frac{3x+4}{(x+1)(x+2)} \equiv \frac{1}{x+1}+\frac{2}{x+2}$$

Partial fractions is a method that allows a rational function to be decomposed into simpler expressions. This is often done to assist in some other process, e.g. differentiation or integration.

Link to Simple Deductive Proof SL 1.6

Link to Partial Fractions to rearrange the integrand AHL 5.15

Example 1.4.5

Express $\frac{5x+16}{x^2+7x+10}$ in partial fractions.

Solution

$x^2+7x+10=(x+2)(x+5)$

Setting $\frac{5x+16}{(x+2)(x+5)} \equiv \frac{A}{x+2}+\frac{B}{x+5}$ for constants A and B gives

$5x+16 \equiv A(x+5)+B(x+2)$

Method 1: Equating the coefficients of the linear functions on both sides gives $5 = A + B$, $16 = 5A + 2B \Rightarrow A = 2, B = 3$

Method 2: Since the identity above is true for all values of x it will be true for any particular values of x.

Putting $x=-5 \Rightarrow B=3$

Putting $x=-2 \Rightarrow A=2$

Concluding $\frac{5x+16}{x^2+7x+10} \equiv \frac{2}{x+2}+\frac{3}{x+5}$

Assessment tip

The syllabus states that in IB examples there will be a maximum of two linear terms in the denominator and the degree of the numerator will be less that the degree of the denominator.

Since in general you need as many constants as the degree of the denominator, in these examples you will need an A and a B.

Note

Method 1 is more rigorous and the equating of coefficients would show if the proposed representation were impossible. Method 2 is often quicker to use if you are sure that the rational function can be split in this way.

Example 1.4.6

(a) Express $\frac{1}{r^2+r}$ in partial fractions.

(b) Hence find an expression, in terms of n, for the sum $\sum_{r=1}^{n}\frac{1}{r^2+r}$

Solution

(a) $r^2+r=r(r+1)$ $\qquad \frac{1}{r(r+1)} \equiv \frac{A}{r}+\frac{B}{r+1} \Rightarrow 1 \equiv A(r+1)+Br$

So $A=1, B=-1$ $\qquad \frac{1}{r^2+r} \equiv \frac{1}{r}-\frac{1}{r+1}$

(b) $\sum_{r=1}^{n}\frac{1}{r^2+r}=\sum_{r=1}^{n}\frac{1}{r}-\frac{1}{r+1}=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n}-\frac{1}{n+1}$

$=1-\frac{1}{n+1}=\frac{n}{n+1}$

Note

Example 1.4.6 illustrates how partial fractions could be used to create a "telescoping" series.

Proof by induction

Proof by induction is the most formal proof that you will be expected to do. There is a set way of laying out the proof to comply with the rigorous logic of the proof. Proof by induction is used to show that a statement $P(n)$ is true for all $n \in \mathbb{Z}^+$. Essentially, we have an infinite number of statements that we have to prove.

This is how you lay out an induction proof.

Identify the statement $P(n)$ that you intend to prove, for all $n \in \mathbb{Z}^+$.

Prove that $P(1)$ is true.

Assume $P(k)$ is true and show that this implies that $P(k+1)$ is true. This is called the induction step.

Conclude with the following comment "$P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$"

This standard final comment needs to be learned by heart.

Assessment tip

You must first show that $P(1)$ is true no matter how trivial it appears to be.

The algebra of the induction step working out well should give you confidence that you are doing the process correctly. Remember that you cannot use the statement that you are trying to prove. However, you can keep an eye on it and work towards it. Do not worry if expressions become longer before eventually simplifying.

There will be a reasoning mark for the concluding comment (provided that enough marks have been gained elsewhere) so remember it and give it.

Example 1.4.7

Prove by induction that $\sum_{r=1}^{n} \frac{1}{r^2+r} = \frac{n}{n+1}$, for all $n \in \mathbb{Z}^+$

Solution

Let $P(n)$ be the statement $\sum_{r=1}^{n} \frac{1}{r^2+r} = \frac{n}{n+1}$

LHS of $P(1)$ is $\frac{1}{1^2+1} = \frac{1}{2}$ RHS of $P(1)$ is $\frac{1}{1+1} = \frac{1}{2}$

So $P(1)$ is true.

Assume $P(k)$ is true so $\sum_{r=1}^{k} \frac{1}{r^2+r} = \frac{k}{k+1}$

LHS of $P(k+1)$ is $\sum_{r=1}^{k+1} \frac{1}{r^2+r} = \sum_{r=1}^{k} \frac{1}{r^2+r} + \frac{1}{(k+1)^2+(k+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{(k+1)}{(k+1)+1}$

which is the RHS of $P(k+1)$, as required.

$P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$

Assessment tip

If the question states "prove by induction" then you must do it this way, even if there are other methods of proof. If it just says "prove" and you decide to use induction, state that you are doing so.

For induction to be used, variable n must be a natural number and the statement that you wish to prove must be fully known. You might be asked to do an investigation and then generalize your results into a conjecture which you are then asked to prove by induction.

Example 1.4.8

Prove by induction that $17^n - 1$ is always exactly divisible by 8, for all $n \in \mathbb{Z}^+$.

Solution

Let $P(n)$ be the statement "8 exactly divides $17^n - 1$"

$17^1 - 1 = 16 = 8 \times 2$ so $P(1)$ is true.

Assume $P(k)$ is true, so $17^k - 1 = 8s$ for $s \in \mathbb{Z}^+$

$17^{k+1} - 1 = 17 \times 17^k - 1 = 17\,(8s+1) - 1 = 8 \times 17s + 16 = 8\,(17s+2)$

$17s + 2 \in \mathbb{Z}^+$ so 8 exactly divides $17^{k+1} - 1$, showing that $P(k+1)$ is true.

$P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

SAMPLE STUDENT ANSWER

Prove by induction that $\sum_{i=1}^{n} i^2 + i = \frac{1}{3}n(n+1)(n+2)$, for all $n \in \mathbb{Z}^+$

For $n = 1$ $\quad 1^2 + 1 = \frac{1}{3} \times 1 \times 2 \times 3 \Rightarrow 2 = 2$ ✓

Assume the result for k $\quad \sum_{i=1}^{k} i^2 + i = \frac{1}{3}k(k+1)(k+2)$

So $\sum_{i=1}^{k+1} i^2 + i = \frac{1}{3}(k+1)(k+1+1)(k+1+2)$

$= \frac{1}{3}(k+1)(k+2)(k+3)$

So it is true.

▲ The student knew they should first look at the case $n = 1$ and then assume the result for $n = k$ and attempt to prove it for $n = k + 1$.

▼ The logic of the case $n = 1$ is poor, starting with what is to be proved and putting "tick". The induction step has not been done. You certainly cannot just substitute $k + 1$ for k, they are consecutive integers. The standard concluding sentence has not been given, and even if it had been, there were not enough marks elsewhere for it to have gained the reasoning mark.

The answer above could have achieved 2/8 marks.

The proof should have been as follows:

Let $P(n)$ be the statement $\sum_{i=1}^{n} i^2 + i = \frac{1}{3}n(n+1)(n+2)$

LHS of $P(1)$ is $1^2 + 1 = 2$. RHS of $P(1)$ is $\frac{1}{3} \times 1 \times 2 \times 3 = 2$

So $P(1)$ is true.

Assume $P(k)$ is true, so $\sum_{i=1}^{k} i^2 + i = \frac{1}{3}k(k+1)(k+2)$

LHS of $P(k+1)$ is

$$\sum_{i=1}^{k+1} i^2 + i = \sum_{i=1}^{k}(i^2 + i) + (k+1)^2 + (k+1) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$

$$= \frac{1}{3}(k+1)(k+2)(k+3) = \frac{1}{3}(k+1)(k+1+1)(k+1+2) = \text{RHS of } P(k+1)$$

$P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$

Assessment tip

When dealing with induction on series (like the question here), in the induction step always look to see if the expression contains common factors (especially if you can see that they should be in the expression that you are working towards), rather than multiplying everything out.

Note

It is possible to have a variation to the standard induction proof where the first step is prove the result for $d \in \mathbb{Z}$ rather than for $n = 1$. The induction step would be the same. This would prove the result for all $n \in \mathbb{Z}^+, n \geq d$.

Link to the Binomial Theorem, as proof by induction can be used to prove this theorem utilising ${}^{k}C_{r-1} + {}^{k}C_{r} = {}^{k+1}C_{r}$

Counterexamples

To prove that a statement is not always true, it is sufficient to give just one example when the statement is not true. Such an example is called a counterexample.

Example 1.4.9

Show that the statement "11 exactly divides $n^{10} - 1$, for all $n \in \mathbb{Z}^+$" is false.

Solution

When $n = 11$, $\frac{11^{10} - 1}{11} = 11^9 - \frac{1}{11}$ which is not an integer.

So $n = 11$ is a counterexample.

Note

In fact, in Example 1.4.9, $n = 11$ is the smallest counterexample that could be found.

Example 1.4.10

Find a counterexample to the statement "n exactly divides $(n-1)!+1$ or n exactly divides $(n-1)!$, for all $n \in \mathbb{Z}^+$."

Solution

For $n=4$, $3!+1=7$ and $3!=6$, neither of which is exactly divisible by 4. So $n=4$ is a counterexample.

Note

In fact, $n=4$ is the only possible counterexample in Example 1.4.10

Proof by contradiction

The layout of a proof by contradiction is as follows. You are given a statement to prove. You assume it to be false and proceed to make logical deductions based on that assumption. If you then obtain a result that you know to be impossible, you can conclude that the original statement must be true.

This is because the original statement must be either true or false. If assuming it to be false leads to a contradiction, you can conclude that it must be true.

Example 1.4.11

Prove by contradiction that $\sqrt{3} \notin \mathbb{Q}$.

Solution

Suppose that $\sqrt{3} \in \mathbb{Q}$. Then $\sqrt{3}$ can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$. Furthermore, it can be assumed that a and b are coprime; there is not a prime number that exactly divides both a and b (as they could have been cancelled out). Then

$\sqrt{3} = \frac{a}{b} \Rightarrow 3b^2 = a^2$, which implies that 3 exactly divides a^2, which also implies that 3 exactly divides a (since 3 is a prime). Let $a = 3A$, where $A \in \mathbb{Z}$, then $3b^2 = (3A)^2 \Rightarrow b^2 = 3A^2$. This implies that 3 exactly divides b^2, which also implies that 3 exactly divides b (since 3 is a prime). This gives the required contradiction as we have shown that 3 exactly divides both a and b, but we assumed that a and b were coprime. Therefore, we conclude that $\sqrt{3} \notin \mathbb{Q}$.

Assessment tip

If the question asks for "Proof by contradiction", then you must do it that way. If a question just asks for a proof and you decide to use proof by contradiction, then state that this is the method that you are using.

Example 1.4.12

Let $a + b\sqrt{3} = c + d\sqrt{3}$, where $a, b, c, d \in \mathbb{Q}$.
Prove that $b = d$ and hence that $a = c$.

Solution

Using proof by contradiction, assume that $b \neq d$.
Then $a - c = (d - b)\sqrt{3} \Rightarrow \sqrt{3} = \frac{a-c}{d-b}$

But $\frac{a-c}{d-b} \in \mathbb{Q}$. This is a contradiction. Therefore, we can conclude that $b = d$. Which also gives $a + b\sqrt{3} = c + b\sqrt{3}$ and hence $a = c$.

Solution of systems of linear equations

The equation $ax + by = c$ represents a straight line in 2 dimensions and the equation $ax + by + cz = d$ represents a plane in 3 dimensions. Solving a system of linear equations of this form simultaneously can be thought of geometrically as finding the points where these objects intersect. There could be no solutions, one unique solution or an infinite number of solutions.

Link to AHL 3.18

Assessment tip

The method of solution will depend on whether it is a Calculator paper or a Non-calculator paper.

With a Calculator paper, the easiest method is to use the simultaneous equation solver but you could find the intersection of lines by drawing a graph if you were only working in two dimensions.

With a Non-calculator paper you would reduce the number of equations by eliminating a variable and then repeat this process.

Example 1.4.13

Solve these simultaneous equations:

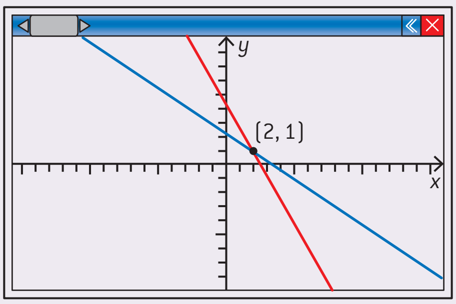
$$\begin{cases} 3x+4y=10 \\ 2x+y=5 \end{cases}$$

Solution

Calculator:

Using the simultaneous equations solver: $x=2$, $y=1$

Or, rearranging: $y=\frac{-3}{4}x+\frac{10}{4}$, $y=-2x+5$ and graphing to find the intersection.



Non-calculator:

(2 × first equation) – (3 × second equation) $\Rightarrow 5y=5 \Rightarrow y=1 \Rightarrow x=2$

The calculator will do the (reduced) row echelon form method for you, but it can also be done by hand on a Non-calculator paper. This is the same as solving the equations simultaneously, and it is useful to learn. A shorthand notation is used with the x, y, z, not being written and the equals signs being represented by a vertical line. A row (representing an equation) can be multiplied by a non-zero constant and multiples of one row can be subtracted or added to another row.

Example 1.4.14

Solve these simultaneous equations:

$$\begin{cases} x+3y+z=9 \\ 2x+y+4z=21 \\ x+5y-3z=-5 \end{cases}$$

Assessment tip

In IB questions you will have a maximum of three equations in three unknowns.

Assessment tip

Putting the description of what has been done with the rows as shown in Example 1.4.14 is good practice and makes it easier to follow your method.

To obtain row echelon form you are attempting to make the start of your notation look as much like

$$\left(\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{array}\right)$$

as possible.

To obtain reduced row echelon form you are attempting to make the start of your notation look as much like

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right)$$

as possible.

Solution (Non-calculator)

Representing the equations by $\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 2 & 1 & 4 & 21 \\ 1 & 5 & -3 & -5 \end{array}$

(Calculations go across the page, then down)

$$\begin{array}{l} \\ row2-2\,row1 \\ row3-row1 \end{array} \begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -5 & 2 & 3 \\ 0 & 2 & -4 & -14 \end{array} \qquad \begin{array}{l} \\ \frac{-1}{5}row2 \\ \\ \end{array} \begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & \frac{-2}{5} & \frac{-13}{5} \\ 0 & 2 & -4 & -14 \end{array}$$

$$\begin{array}{l} \\ \\ row3-2\,row2 \end{array} \begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} \\ 0 & 0 & \frac{-16}{5} & \frac{-64}{5} \end{array} \qquad \begin{array}{l} \\ \\ \frac{-5}{16}row3 \end{array} \begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} \\ 0 & 0 & 1 & 4 \end{array}$$

Either: converting back into equations $z = 4$

$y - \frac{2}{5}z = \frac{-3}{15} \Rightarrow y = \frac{-3}{5} + \frac{8}{5} = 1 \qquad x + 3y + z = 9 \Rightarrow x = 9 - 3 - 4 = 2.$

Or: continue reducing rows in matrix form

$$\begin{array}{l} row1-3\,row2 \\ \\ \\ \end{array} \begin{array}{ccc|c} 1 & 0 & \frac{11}{5} & \frac{54}{5} \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} \\ 0 & 0 & 1 & 4 \end{array}$$

$$\begin{array}{l} row\ 1-\frac{11}{5}\,row3 \\ row\ 2+\frac{2}{5}\,row3 \\ \\ \end{array} \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}$$

Reading off $x = 2$, $y = 1$, $z = 4$

Note

A way to ensure that a calculator cannot be used is to include a parameter as demonstrated in Example 1.4.15

Example 1.4.15

(a) Find the value of λ for which the following system of equations is consistent (this means that there is at least one solution).

$$\begin{cases} x + 3y + z = 9 \\ 2x + 7y + 4z = 21 \\ 4x + 13y + 6z = \lambda \end{cases}$$

(b) For the value of λ found in part (a), find the solutions to this system of equations.

Solution

(a) $\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 2 & 7 & 4 & 21 \\ 4 & 13 & 5 & \lambda \end{array} \qquad \begin{array}{l} \\ row2-2\,row1 \\ row3-4\,row1 \end{array} \begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & \lambda-36 \end{array}$

$\begin{array}{l} \\ \\ row\ 3-row\ 2 \end{array} \begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & \lambda-39 \end{array}$ Last line is $0x + 0y + 0z = \lambda - 39$

So to be consistent $\lambda = 39$

(b) So $z = z$, $y + 2z = 3 \Rightarrow y = 3 - 2z$,

$x + 3y + z = 9 \Rightarrow x = 9 - 3\,(3 - 2z) - z = 5z$

Solutions are of the form $\frac{x}{5} = \frac{y-3}{-2} = z$

Link to AHL 3.14 and AHL 3.15, the solution given in Example 1.4.15 (b) is a straight line through the point $(0, 3, 0)$ and parallel to the vector $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$